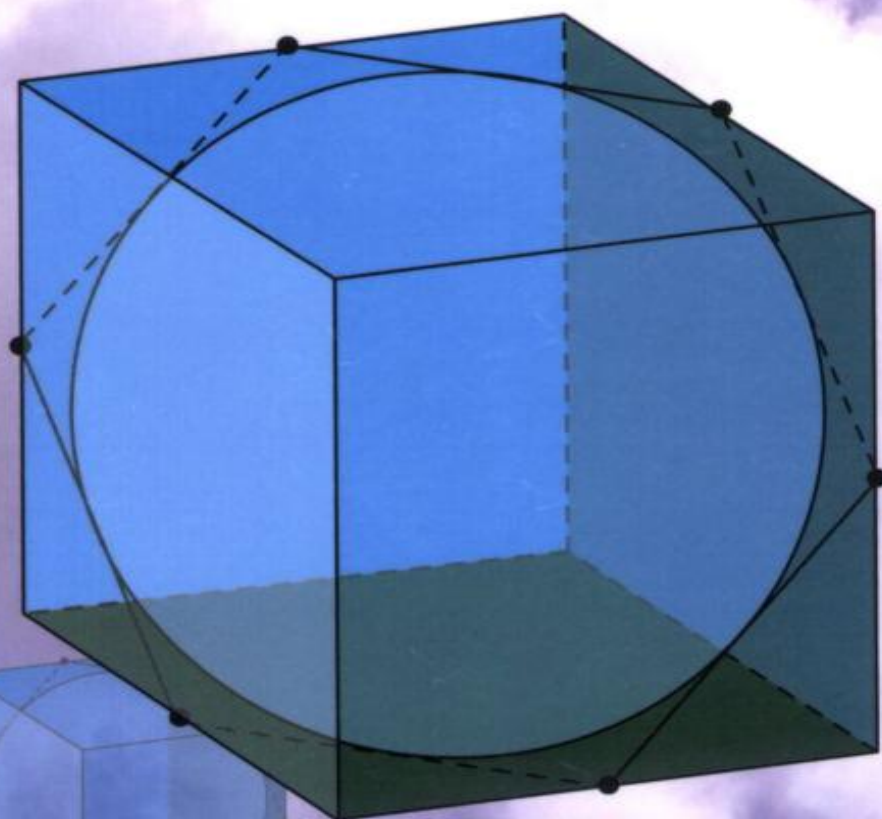


Circle in a Box



Sam Vandervelde



Circle in a Box

Sam Vandervelde



Advisory Board for the MSRI/Mathematical Circles Library

Zuming Feng	Alexander Shen
Tony Gardiner	Tatiana Shubin (Chair)
Kiran Kedlaya	Zvezdelina Stankova
Nikolaj N. Konstantinov	Ravi Vakil
Silvio Levy	Ivan Yashchenko
Walter Mientka	Paul Zeitz
Bjorn Poonen	Joshua Zucker

With thanks to Dan Freed and Mark Kleiman for their support.
Mathematics brought them together as friends when as teenagers
they met at the Math Olympiad Training Program.

The cover art shows how to inscribe a circle of maximal diameter in a unit cube. The inscribed circle, which touches the cube at six points, has diameter of length $\sqrt{6}/2$.

2000 *Mathematics Subject Classification*. Primary 97D50; Secondary 97U40, 97A20.

For additional information and updates on this book, visit
www.ams.org/bookpages/mcl-2

Library of Congress Cataloging-in-Publication Data

Vandervelde, Sam

Circle in a box / Sam Vandervelde.

p. cm. — (MSRI mathematical circles library ; 2)

ISBN 978-0-8218-4752-7 (alk. paper)

1. Mathematics—Study and teaching (Secondary)—Handbooks, manuals, etc. I. Mathematical Sciences Research Institute (Berkeley, Calif.) II. Title.

QA11.2.V355 2009

510.71'2—dc22

2008051596

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to reprint-permission@ams.org.

© 2009 by Mathematical Sciences Research Institute. All rights reserved.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

Visit the MSRI home page at <http://www.msri.org/>

10 9 8 7 6 5 4 3 2 1 14 13 12 11 10 09

Preface

I never intended to write a handbook on math circles. I was simply delighted that circumstances should conspire that I was able to organize the Stanford Math Circle in the summer of 2005 and to have the opportunity to lead around half of the sessions during its first year. I thoroughly enjoyed the enterprise and felt that it was quite successful as a means for presenting great mathematics to high school students in the area. But as they say, no good deed goes unpunished.

The following spring Paul Zeitz, the director of the San Francisco Math Circle, mentioned that the Mathematical Sciences Research Institute (MSRI) intended to assemble a resource for individuals who were interested in launching a math circle at their institution. It would be a complete package containing advice on creating and sustaining a math circle, tips on how to effectively lead a math circle, some sample presentations, a DVD that would highlight math circles in action, a companion web site to supplement the handbook, and more. A math circle in a box, as the project came to be dubbed.

The first thought that came to mind was, “That’s a wonderful idea; I would love to receive a copy of those materials when they are finished.” However, I could not suppress my enthusiasm for becoming involved in such a project and very shortly found myself faced with the daunting task of creating the handbook now before you. In the process I discovered that I had more to say about organizing and leading math circles than I realized. But I have also learned an enormous amount from my colleagues around the country who have been running these mathematical outreach events for much longer than myself. I am grateful to all those individuals who took the time to share their accumulated wisdom with me—summaries of their accounts appear as “Circle Snapshots” sprinkled throughout the text, and their

collected knowledge is incorporated into practically every single page. Besides the individuals featured in the snapshots I also had very fruitful and enlightening conversations with Mark Saul, Scott Carter, Dan Silver, Cornelius Pillen, Jennifer Jeffrey, Sharon Madison, and Steven Krantz.

Of course I am indebted to Paul Zeitz for recommending the project to me in the first place. I would also like to acknowledge the steadfast guidance and encouragement from the people at MSRI who conceived of and followed through with this project: David Eisenbud, Hugo Rossi, Jim Sotiros, and Kathleen O'Hara. (It was Hugo Rossi, I believe, who originally coined the phrase "circle in a box.") Jim Sotiros also deserves a great deal of credit for sharing his expertise on raising funds; he contributed several sections on the subject towards the end of Chapter 2. The author-support team at the AMS had many helpful suggestions as well; in particular, the "Circle Snapshots" pages are far more effective now than they were originally. I greatly appreciate their advice.

I was encouraged by the fact that many fellow math circle enthusiasts were eager (or at least willing) to read through the preliminary manuscript and offer helpful feedback. My thanks to Tatiana Shubin, Zvezdelina Stankova, Jim Sotiros, and Matt Beck for taking the time to catch typos and offer comments. I am especially indebted to Hugo Rossi for his comprehensive, detailed, and insightful remarks. His suggestions led to many substantial revisions; the handbook was markedly improved as a result.

Finally, if one wishes to truly experience and appreciate the vitality of a math circle, there is no substitute for seeing one in action and hearing the students and mathematicians involved speak about them firsthand. For this reason MSRI has undertaken a project to document math circles. The result of their efforts is the wonderful DVD *Within the Circle*, available upon request from MSRI. For further information contact them by writing to info@mathcircles.org, calling 510-642-0143, or visiting their web site devoted to math circles, www.mathcircles.org.

Contents

I	Oversight	7
1	Molding a Math Circle	9
1.1	What is a mathematical circle?	9
1.2	Origins	11
1.3	Knowing your audience	13
1.4	Location, location, location	16
1.5	Setting the schedule	17
1.6	Filling the schedule	20
1.7	Getting the word out	23
2	Supporting a Math Circle	29
2.1	Delegation and collaboration	29
2.2	Embracing technology	32
2.3	Money matters	36
2.4	Finding funding	41
2.5	Granted	43
2.6	Countdown to a math circle	46
3	Sustaining a Math Circle	47
3.1	Retaining students	47
3.2	The Fountain of Youth	50
3.3	MyMathCircle	54
3.4	Paperwork	56
3.5	How am I doing?	58
3.6	Paging all parents	59
3.7	Going the distance	62
4	Leading a Math Circle	65
4.1	Premium presentations	65
4.2	Transcendent topics	68

II	Presentations	75
5	The Game of Criss-Cross	77
6	Double Time	83
7	King Chickens	89
8	Into the Unknown	95
9	Sneaky Segments	101
10	Heads or Tails	107
11	Circling the Square	113
12	Making Change	121
13	Two Squares Are Better Than One	129
14	Reflection in a Circle	137
III	Appendices	147
A	Sample Documents	149
B	Warm-ups	159
C	Sample Grant Proposal	165
D	Sample Grant Report	171

Part I

Oversight

Chapter 1

Molding a Math Circle

1.1 What is a mathematical circle?

On May 17, 2006, math department chairs from colleges and universities across the greater New York City area attended an organizational meeting held at the Courant Institute of Mathematical Sciences to take part in a mock math circle meeting and consider the possibility of starting one at their own institution. The event was organized by Mark Saul, a former administrator at the National Science Foundation and a long-time advocate of math circles. In a letter sent to departments announcing the meeting, Mark described math circles in this way:

Mathematical circles are a form of outreach that bring mathematicians into direct contact with pre-college students. These students, and sometimes their teachers, meet with a mathematician or graduate student in an informal setting, after school or on weekends, to work on interesting problems or topics in mathematics. The goal is to get the students excited about the mathematics they are learning; to give them a setting that encourages them to become passionate about mathematics.

Defining a math circle

Math circles can have a variety of styles. Some are very informal, with the learning proceeding through games, stories, or hands-on activities. Others are more traditional enrichment classes, but without formal examinations. Some have a strong emphasis on preparing for olympiad competitions; some avoid competition as much as possible. Models can use any combination of these techniques, depending on the audience, the mathematician, and the environment of the circle. Athletes have sports teams through which to deepen their involvement with sports; math circles can play a similar role for kids who

What all math circles have in common

like to think. One feature all math circles have in common is that they are composed of students who enjoy learning mathematics, and the circle gives them a social context in which to do so.

*The purpose of
Circle-in-a-Box*

The discussion, advice, and vignettes contained in these pages will convey a fairly complete picture of math circles. By the end most readers should be convinced that a math circle is a good idea for their institution, students, or child, depending upon their perspective. But that is not the true purpose of this document. More than just being informative, it is meant to function as a resource for individuals who are willing to take the leap of faith and actually initiate and maintain a math circle within their own community. These directors and coordinators, some of whom are featured in the “Circle Snapshots” sprinkled throughout the text, form another common thread among all math circles. Each person has a unique organizational style, but they all share the belief that presenting beautiful mathematics to motivated students in an engaging manner is a worthwhile, important enterprise.

There are as many compelling stories behind the launching of math circles as there are organizers. Perhaps a faculty member at a college or university owes their interest in mathematics to a circle they attended as a secondary student, and now wishes to provide a similar experience for students in their area. Since the math circle phenomenon is a relative newcomer to the

One feature all math circles have in common is that they are composed of students who enjoy learning math, and the circle gives them a social context in which to do so.

*How do math
circles begin?*

United States, these professors tend to have arrived from a country with a rich tradition in math circles, such as Russia or Bulgaria. This was the case with Zvezdelina Stankova and the Berkeley Math Circle, for instance. Or perhaps a parent summons the courage to approach a local math department to forward the idea of partnering to establish a math circle, as occurred when Jennifer Jeffrey contacted Steven Krantz at the University of Washington, St. Louis. Bob and Ellen Kaplan, who both taught in the Boston area, became so tired of the negative attitude towards math ingrained in their students that they invited a group to meet in their living room to discover the true beauty of the subject. This informal gathering eventually grew to become The Math Circle.

A vibrant math circle can be a source of great inspiration to students and a rewarding enterprise for the mathematicians coordinating and leading them. However, just as with any sort of community, there is more to establishing a thriving math circle than meets the eye. Introducing students to exciting, accessible mathematics in a creative, engaging manner is certainly a key element. For this reason the entire second part of this handbook is devoted to laying out a

wide variety of sample presentations, with a range of topics and difficulty levels, along with detailed presentation notes for the instructor and carefully developed problems and solutions. However, while solid mathematical content and delivery is certainly necessary for a successful math circle experience, it is not sufficient. Careful thought must also be given to the issues of how to attract and retain students, when and where to meet, how to structure the meeting time, whether to set up a web page for the circle, and more. Following a brief historical interlude, the upcoming sections will explore the various options available and consider how a number of successful math circles have addressed these issues.

Logistics behind a successful math circle

1.2 Origins

Mathematical enrichment activities in the United States have been around for at least thirty years, in the form of residential summer programs, math contests, and local school-based programs. The concept of a math circle, on the other hand, with its emphasis on convening professional mathematicians and secondary school students on a regular basis to solve problems, has appeared only within the past twelve years. This form of mathematical outreach made its way to the U.S. most directly from Russia and Bulgaria, where it has been a fixture of their mathematical culture for decades. (The first ones appeared in Russia during the 1930's; they have existed in Bulgaria for a century.) The tradition arrived with émigrés who had received their inspiration from math circles as teenagers. Many of them successfully climbed the academic ladder to secure

Tracing the roots of math circles

Math circles function as an entry point to school-level mathematical culture in Russia.

positions within universities, and a few pioneers among them decided to initiate math circles within their communities to preserve the tradition which had been so pivotal in their own formation as mathematicians. The Mathematical Sciences Research Institute (MSRI) in Berkeley, California became involved at an early stage by supporting the Berkeley Math Circle. Not long after

Steve Olson highlighted this math circle in his book *Countdown*, since a couple of members of the 2001 U.S. International Mathematical Olympiad (IMO) team attributed their success in part to the problem-solving sessions offered at Berkeley. In this and other ways math circles began to attract national attention as a means for encouraging students to enjoy, explore, and excel in mathematics.

The author was part of a delegation from the United States that recently visited Russia to observe the secondary school math culture in Moscow and St. Petersburg first-hand, including visits to a number of active math circles. These circles constitute only one piece of a wide array of programs aimed at identifying

*The role of math
circles in Russia*

and cultivating mathematical talent amongst the Russian student population, at least those privileged enough to live near an urban center with a university or institute supporting such activities. Russian students grow up within an educational system that promotes problem solving (as opposed to computation or textbook exercises) to a much greater degree than in the United States. Therefore it is a natural step for these students to begin taking olympiads and attending math circles during their middle school years. (In contrast to a math contest, an olympiad typically asks students for an explanation rather than a computation. These olympiads are offered at all levels and geographic scales; the most prestigious is the All-Russian Olympiad which helps to determine their IMO team. It is the Russian analogue of the USA Math Olympiad.) The olympiads and circles feed into one another and serve as the entry point to the school-level mathematical culture in Russia. From there the stronger students might be admitted to one of several “specialized schools” with advanced math classes taught by teachers whose mathematical pedigree is comparable to those at the best American high schools. Students who wish to pursue mathematics may also attend a selection of summer programs and participate in various special events such as “math battles” or “math fests.”

*Connections
between schools
and universities*

It is illuminating to contrast this system with that currently in place in the United States. The most notable difference concerns the general perception of mathematics as a domain for investigation versus a tool for science or business. This is due in large part to the significantly higher level of interaction between research mathematicians and secondary school students in both Russia and Bulgaria. It is not uncommon for graduates with degrees in mathematics to enter the teaching work force as well as to continue active research at some level. It is also standard for well-known mathematical figures to join the staff at summer programs. The Moscow Center for Continuous Mathematics Education, which coordinates a wide array of activities for school-level students, is housed in the same building as the Independent University of Moscow and has ties with Moscow State University as well. Most striking, perhaps, is the incredible level of involvement of a considerable number of undergraduate and graduate students who spend hours each week helping out at math circle meetings, grading olympiads, or otherwise supporting the effort. Because of their faithfulness in repaying their debt to the prior generation of university students, kids attending math circles receive an unparalleled amount of personal attention. It is standard operating procedure to devote the bulk of a meeting

While solid mathematical content and delivery is certainly necessary for a successful math circle experience, it is not sufficient.

to individual discussions with mentors, in which participants present their solutions to problems which had been distributed during the previous session.

At the same time, Russian mathematicians working to encourage students with an interest in math face many of the same challenges that their American counterparts do. There are state-mandated multiple choice tests for all students intended to guarantee minimal competency in math which divert attention from high performers. Those who conduct mathematical research often find themselves at odds with those who study mathematics education. Many students who would benefit greatly from attending math circles are geographically out of reach. Time and funding for worthwhile projects is often in short supply. But it is heartening to note that because of these similarities each nation can learn from successful approaches discovered and employed by the other.

*Common
challenges*

1.3 Knowing your audience

By far the most important matter that the organizer of a new math circle must settle concerns its intended participants, as this decision will inform choices to be made on all the other issues raised in the upcoming sections. The arrangement

The Math Circle in Boston reaches students as young as four years old.

that naturally springs to mind, and hence the most common model, is to set up a math circle targeting motivated students interested in math within a certain geographical area. In this case the common denominator among members of the audience is each individual's (or their parent's) knowledge of and desire to attend

the circle, independent of school affiliation or other background. Although this is likely to be the appropriate choice in most cases, there are other possible models for math circles that target teachers or underserved populations. While they are not addressed separately here, information pertaining to these models appears throughout the text. In particular, the San Francisco Math Circle snapshot and the sample grant proposal in the appendices contain helpful ideas regarding math circles for these intended audiences.

*Different models
for math circles*

For most new coordinators, the first truly substantial decision concerns the level at which to pitch the math circle. Given the organization of grades in the United States, the natural options to consider are the middle school level, high school level, one size fits all, or two or more levels offered simultaneously. (Although The Math Circle in Boston reaches students as young as four years old.) The final option is more ambitious than is advisable for a new circle, but it is a healthy development among circles which outgrow their original charters. In terms of content, material intended for middle school students may

◦ ◦ ◦ ◦ ◦ ◦ ◦
◦

Circle Snapshots



Name: The San Francisco Math Circle
Location: San Francisco State University
Director: Paul Zeitz
Email: zeitz@usfca.edu
Meeting time: Mondays 4:30–5:30pm
Web site: www.sfmathcircle.org

◦ ◦ ◦ ◦



◦
◦
◦
◦
◦
◦

Paul Zeitz has felt called to provide mathematical opportunities to “unenriched kids” for a long time, due in no small part to the fact that he grew up fatherless in a gritty New York neighborhood himself. He likes to describe the summer he participated in the very first Math Olympiad Program, when he was surrounded by teenagers from “exotic two-parent suburban families.” As a co-director of the Bay Area Math Olympiad, he was mildly distressed to notice that none of the participants (much less high scorers) were located in San Francisco. So when MSRI approached Paul the winter before his sabbatical year with an idea for presenting exciting mathematics to city kids and teachers, Paul was eager to help head up the new program.

The pivotal idea about which the San Francisco Math Circle revolves is the premise that the most effective way to reach kids who would otherwise never hear about a math circle is to set up an excellent program for teachers and compensate them for bringing their students along. Teachers enjoy a quality mathematical experience, receive continuing education credit for regular participation, and earn \$100 per week for transporting their students to the event. Another central tenant concerns the importance of establishing a relationship between the instructors and the audience. For this reason Paul asks his presenters to commit to four or five weeks and chooses a common theme which all the groups investigate during that time. His philosophy seems to be working—on a typical Monday between fifteen and twenty teachers bring a diverse crowd of students, upwards of eighty kids each week.

Several factors have made this program as successful as it has become. According to Paul, finding their present location at SFSU was one of the keys, since this site is much more accessible to his target population. Teaming up with Matthias Beck from the outset also made a huge difference. Paul notes that Marianne Smith deserves an substantial amount of the credit for helping to put the organizers in touch with a group of dedicated teachers and key administrators during the months leading up to their first session, a feat which is notoriously difficult to pull off. Finally, none of this would be possible without significant financial backing, which was discovered through and is managed by the Mathematical Sciences Research Institute.

assume basic algebra such as factoring and the quadratic equation, techniques for solving simple word problems, standard area and volume formulas from geometry, sequences or patterns, and similar topics. High school students will have also been exposed to Euclidean geometry, polynomials, exponential and logarithmic functions, trigonometry, elementary counting techniques, and other precalculus topics. Therefore middle school students would certainly appreciate

There is a force at work that has the effect of lowering the average age of a math circle over the years.

presentations introducing graph theory, basic probability, iterated functions, elementary number theory, and so on. However, more than likely they would become quickly lost if this material were presupposed. Topics that are better suited for a high school group might include combinatorial proofs of Fibonacci identities, Gaussian integers and sums of two squares, inversion in Euclidean geometry, complementary sequences, and

a host of other exciting mathematics. Further thoughts on selecting topics are assembled in the chapter on leading math circles.

The range of mathematics available to high school students, as opposed to middle school students, turns out to be disproportionately broader than a few extra grade levels of mastered curriculum might suggest. There is no denying that older kids have a greater base of knowledge, which permits a wider variety of potential topics. But there is another more significant sociological factor at work. If a high schooler attends a math circle it is usually because that student has answered the question, “Am I interested in and good at math?” in the affirmative, has independently studied advanced mathematics, and is likely to be substantially beyond the average high school student in terms of mathematical experience. On the other hand, late elementary and middle school students (and their parents) are usually in the midst of exploring this question, and attendance at the local math circle can be part of the process of finding the answer. These students have not had nearly as many years in which they knew they wanted to focus on mathematical thinking. The point is, a high school group is usually further above grade level in mathematical experience, on average, than a middle school group. This fact should not preclude the presentation of challenging material that will stretch middle school students’ minds; presumably they are attending the circle because they are interested in discovering new mathematical horizons, even if they do not completely understand the material on the first pass. But it is a good idea to keep this distinction in mind.

The rich variety of interesting topics available at the high school level would seem to make this age group far more desirable, especially to a college or university math department. Therefore it may come as a surprise to learn that there

Topics for middle and high school students

The difference between middle and high school students

*The Fountain of
Youth effect*

are very few math circles which primarily serve high school students. (The Mobile Math Circle is one of the exceptions.) There is a counterintuitive force at work that has the effect of lowering the average age of a math circle over the years. This phenomenon will be discussed in more detail when dealing with the issue of retaining students, but it is important to note at this stage that a math circle which intends to target older students should be prepared to actively resist this tendency. For the same reason, organizers who decide to make their circle accessible to all age levels should commence such an undertaking with realistic expectations; practically every circle that begins in this fashion metamorphoses into a middle school math circle soon after.

1.4 Location, location, location

*Choosing a site
for math circle
meetings*

Selecting a suitable meeting site is the next major decision to be made. Faculty members at a college or university will probably opt to host the circle in a classroom or auditorium regularly used by their math department. For secondary school teachers and parents the decision is less obvious. Although it may require more legwork, finding a math department at a local institution that is willing to sponsor and host the math circle has several strong advantages over running it out of a school, workplace, or home. For starters, a college or university provides a neutral, well-known, scholarly setting for the pursuit of mathematics. Besides, students tend to be excited by the prospect of spending time on a university campus. When the Peninsula Math Circle relocated to Stanford University after one semester of operation, its enrollment immediately doubled. Furthermore, corporations or private schools run the risk of having their mathematical outreach misconstrued as a recruitment effort. (Unless, of course, this is the intent.) Departmental sponsorship has the added advantage that finding speakers, which is one of the main challenges in operating a math circle, becomes much more feasible. At the end of the day there may be other constraints which make this option less appealing, but it is usually worth investigating.

“The [Mobile] Math Circle has been by far the most effective outreach activity we have ever undertaken.”

*Obtaining the
support of a local
math department*

Here are a few thoughts for organizers without a university affiliation who wish to enlist the support of math faculty at a nearby institution. If possible, avoid introducing new projects to faculty members during the summer, when they are typically at conferences, involved in summer programs, engaged in research, or writing their books; or in the fall, when they have their hands full with the start of classes and other responsibilities. One is more likely to receive

It goes without saying that Friday night is not desirable, but Monday night works better than one would at first expect.

a warm reception in the winter or spring when there is time to plan ahead for a new departmental activity. Secondly, contact the chair of the department; this person should be kept apprised of the math circle and will know who in the department is most likely to be willing to help out. Mention the name of another person on the faculty if someone else is already interested in coordinating the math circle. Finally, mention that such a project would be beneficial to the department in addition to providing an exciting mathematical opportunity to students in the area. Dan Silver at the University of Southern Alabama (USA) stated that, “The [Mobile] Math Circle has been by far the most effective outreach activity we have ever undertaken.” Part of the original motivation for their

Benefits of a math circle to a department

math circle was to draw students into the math major at USA; the strategy worked even better than they had hoped. It also had the effect of integrating the mathematical community in their region, bringing high school, college, and graduate students together with math faculty.

Secondary school teachers with the right combination of mathematical background, enthusiasm, and access to potential speakers have also managed to successfully conduct math circles at their schools. For instance, a math circle ran at Henry M. Gunn High School in Palo Alto, CA for many years. Making the event welcoming to students outside the school is more of a challenge in this case. Finding qualified people to lead the sessions can also be more difficult. Therefore these groups are often composed solely of students from within the school who spend time preparing for math contests or together poring over books containing mathematical activities or advanced topics. As such they are more aptly described as math clubs than math circles. However, exceptions do exist; the St. Mark’s Institute of Mathematics is a good illustration.

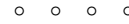
Math circles at high schools

1.5 Setting the schedule

The factors that will influence the choice of times and dates to meet vary so widely from one math circle to the next that it would be impractical to attempt to include them all. Each organizer will need to make allowances for room and speaker availability (don’t compete with the Tuesday afternoon colloquium), local traditions and holidays (Wednesday night is out in the Bible Belt), traffic patterns (nobody would make it to campus on time before 6:30 on weekdays), and a myriad of other factors. However, there are still several general principles that can be applied in a range of settings.



Circle Snapshots



Name: St. Mark's Institute of Mathematics

Location: St. Mark's School, MA

Director: Jim Tanton

Email: mathinstitute@stmarksschool.org

Meeting time: Mondays 4:30–6:30pm

Web site: www.stmarksschool.org/academics/mathinstitute.aspx

Many secondary schools sport a math club, often for the purpose of practicing for and participating in various math contests. Occasionally, however, a school-based activity achieves the purpose and vision normally associated with a math circle: to explore exciting, extracurricular mathematics in an engaging, interactive manner with interested students from the general community. The St. Mark's Institute of Mathematics is a perfect example of such an enterprise.

Jim Tanton, the director of the institute, hales from Australia. He completed his doctoral work at Princeton and began heading down the academic track before his passion for teaching diverted him to spend several years working with the Kaplans at the Boston Math Circle. When St. Mark's School in Southborough, MA invited him to join their faculty and found an Institute of Mathematics, Jim was delighted. The institute, whose mission is to engage in mathematical outreach at all levels, began operation in the fall of 2004.

True to its mission statement, the institute offers a wide variety of mathematical activities and programs. Jim leads two hour-long math sessions on Monday afternoons. The middle school students appear at 4:30, followed by a high school group; in total between twenty and thirty students show up on a typical day. The circle is open to all kids in the area; more often than not the local students outnumber those from St. Mark's. Jim also conducts five core courses for teachers who are working towards a Master of Education degree with a concentration in math. The classes, which are offered through Northeastern University, meet at the Institute on Saturdays or during the summer and provide the primary source of funds for the Institute's activities.

Besides holding forth on his favorite mathematical topics, Jim also loves to write about mathematics. Several years ago he decided that he wanted to publish a monthly newsletter containing interesting problems, engaging exposition, and ideas for further research, all centered about a common theme. This lively document, typically filling the front and back of one sheet of paper (a different color each month), was originally mailed out to math departments around the greater Boston area. The circulation has now increased to around six-hundred addresses, many of them students. To this day nobody has ever requested to be taken off the mailing list.

For starters, the age of the students attending the circle makes a difference: middle school students can squeeze a late afternoon time slot into their weekday schedules more easily than high schoolers, who have a greater number of demands placed on their after-school time. Both age groups can usually manage an early evening meeting time, although this may again be asking more of high school students given their typically larger homework load. It goes without saying that Friday night is not desirable, but Monday night works better than one would at first expect. Gathering on a weekend allows students from a wider geographic area to attend. Thus Saturday morning or Sunday afternoon have

Weekday versus weekend meeting times

It is preferable to shorten the year in favor of regular meetings rather than to thin out a schedule with biweekly math circles.

proven to be a viable alternative for many groups like the Boston Math Circle. (The official title of the latter group is simply “The Math Circle,” but the modifier “Boston” will be inserted from here on to avoid confusion.) A weekend time is especially well-suited for larger groups such as the San Diego Math Circle.

The intended mode of transport will also play an important role in determining a meeting time. The San Francisco Math Circle provides programs for both

teachers and students; since these teachers drive their students to and from the circle a late afternoon time is essentially the only option available. A math circle targeting an urban population that would travel via public transportation might also be better off in the afternoon, when service is more frequent. On the other hand, parents are the primary chauffeurs to the San Jose and Stanford Math Circles, which meet on Wednesday evenings and Sunday afternoons, respectively. To some extent, families will organize their schedules around a math circle they strongly wish to attend; often the best plan is to pick a feasible time and stick with it for a year, knowing that it is impossible to please everyone.

Transportation is a factor

Most organizers make very similar choices with respect to meeting dates. It makes sense to slate the initial meeting a week or two after Labor Day; by this time students have begun to establish a weekly routine but have not yet become too busy to consider a new activity. This time frame also gives coordinators the chance to advertise the circle. Universities on a quarter system may opt for an October start, but should probably begin as early as their academic schedule permits in light of the previous considerations. There is time for a couple of meetings after Thanksgiving, although attendance usually dwindles. Following winter break the default would be to continue meeting until final exams begin to loom for either those attending the circle or those hosting it, which typically occurs in late April or early May. However, there may be good reason to draw the year to a close slightly earlier, such as difficulty in finding enough good

The calendar

speakers to support a full year's worth of meetings, or simply a desire to avoid an overly ambitious schedule the first year. In early April the Stanford Math Circle segues into practice sessions for the American Regions Math League (ARML) team, which take place at the same time and location.

*How often should
a math circle
meet?*

Coordinators of most existing math circles favor weekly meetings, for good reason. There is much to be gained from holding a regular math circle—the event becomes part of a weekly routine and it is possible to establish a sense of continuity. A student who misses one biweekly meeting (even if for a good reason!) goes three weeks without attending the math circle, at which point she decides that she can live without it. The exception to this rule occurs, for example, when a middle school circle gathering every other week is associated with a companion high school circle meeting more regularly, so that students have a fall-back option during the off weeks. But in general it is preferable to shorten the year in favor of regular meetings rather than to thin out a schedule with biweekly math circles.

*How long should
a meeting last?*

Lastly, almost all math circle coordinators make the same choice with regard to the length and format of a meeting. While an hour is sufficient for presenting significant mathematics in a lecture format, it is barely enough time to allow students to make substantial headway in exploring a new topic for themselves and to actually work on problems in the process. On the other hand, two hours begins to approach the limit of a group's attention span, even with a short break in the middle. Besides, it would be asking a lot of a family to devote more time than this to any single activity. Therefore most sessions last from an hour and a half to two hours in duration. Announcements, mathematical tidbits, or speaker introductions usually occupy the first five to ten minutes while those caught in traffic arrive. The presentation for the day follows; the style of delivery should be so interactive that there is no need to reserve time for questions at the end. Instead, save a few minutes for snacks, or provide a ten minute break in the middle of longer sessions.

Recruiting a sufficient number of good speakers to lead the circles is the greatest yearly challenge organizers face.

1.6 Filling the schedule

As all veteran math circle coordinators can relate, recruiting a sufficient number of good speakers to lead the circles is the greatest challenge they face from year to year. Their selection is crucial, for the person presenting the mathematics will determine whether the participants eagerly explore or gradually disengage

from the day's topic. Suffice it to say that the job calls for an understanding of interesting, accessible mathematical topics that are not part of the standard secondary curriculum; knowledge held by individuals who typically pursue careers in academia, software engineering, national security, or other fields requiring specialized math skills. On the other hand, this knowledge could be squandered if it is not presented in a creative, engaging, interactive manner; much as a favorite middle or high school math teacher would have done. Paul Zeitz, the director of the San Francisco Math Circle, has this cautionary advice to offer:

Don't assume that knowing cool math will immediately translate into an exciting, compelling mathematical experience for secondary school students! It is harder to break with the lecture style of presentation than most people realize, and delivery makes a difference.

This being said, there are a few guidelines that will help ensure a solid lineup of speakers for any math circle. To begin, start looking around early—the best people are usually the busiest. Look for people that fall into at least one of the categories mentioned above, then provide resources to give support in the other category. For example, college or university faculty members often have

One of the most popular presentations made at the Stanford Math Circle last year was given by a local private school math teacher.

interesting topics at their fingertips. However, they are accustomed to lecturing to advanced students and may not have as much experience conducting an interactive, open-ended exploration at the secondary school level. Sending out a friendly “what to expect” letter a week or two in advance, which includes a description of the style and content employed at a typical math circle meeting, will go a long way towards bridging the gap and will probably be much appreciated. (A sample email message is contained in the appendices.) It is

also a good idea to ask speakers to prepare a handout ahead of time with a variety of problems related to the presentation. This will help keep students occupied and engaged during the session, and will also provide a gauge of the difficulty level of the topic, in case adjustments need to be made given the background of the audience. Above all, speakers unfamiliar with the math circle philosophy will benefit from seeing one in action, so invite new instructors to visit the circle a week or two in advance of their debut. This practice can become standard protocol: at the Utah Math Circle all first-time speakers are required to attend a math circle before leading a session.

Professors are certainly not the only ones with the requisite mathematical background. Graduate students, high school teachers, and a wide variety of

The greatest challenge to the coordinator

Finding good speakers for a math circle

Tips for inviting college level speakers

*Candidates for
math circle
speakers*

professionals can all make excellent candidates for leading math circles. One of the most popular presentations at the Stanford Math Circle last year was given by a local private school math teacher. In theory undergraduates and middle school teachers could also be included in this list, but these two groups might be uncomfortable at the helm. Undergraduates may be too close in age to their audience, and will probably lack speaking experience. Middle school teachers would be more effective than they might suspect, but may not consider themselves sufficiently advanced in math to feel comfortable leading a mathematical enrichment activity.

While substantial mathematical content should constitute the staple of any math circle diet, there is certainly latitude for incorporating related subjects.

*Qualities of good
speakers*

Regardless of where one looks, the universally accepted method for tracking down math experts can be summarized as: keep an ear to the ground and persistently ask around. It makes sense to start with local colleges or universities, but look further afield also. Keep in mind that the most important qualities for a speaker to possess are a dynamic, engaging presentation style, a love for and general competency in mathematics, and some familiarity with the target age group. If there is money available to support a guest's travel and accommodations, consider inviting a more well-known mathematician to speak at the circle session. Anticipating such a visit can provide focus and generate enthusiasm among regular math circle participants. It might work well to arrange for this guest to also give a department colloquium; the trip becomes twice as worthwhile for the speaker and other funds may become available for covering the visit. Naturally the director of the circle should speak frequently as well, without letting the responsibility of preparing for sessions turn into an unwelcome burden.

*Resources for
speakers*

Most individuals who enjoy mathematics will already have their own list of favorite problems which they would delight in discussing. However, some potential speakers may be unsure as to which topics would be best to present since they do not have regular contact with pre-college students or may not have a working knowledge of mathematics outside the standard secondary curriculum that is both accessible and well-suited for exploration. Look no further—the chapter on leading a math circle includes a long list of possible topics along with suggestions on how to present them. Furthermore, the second part of this handbook contains a variety of ready-made math circle sessions that can be used directly or adapted for a particular audience. And while substantial mathematical content should constitute the staple of any math circle diet, there is certainly latitude for incorporating related subjects into the schedule. Thus students might enjoy learning about mathematics as it is applied to computer

graphics, or sailboat design, or survey data analysis, for example.

Finally, there is the question of how often individuals will visit a particular math circle. Should a speaker come just once? This approach places less of a burden on the speaker and ensures a greater variety of topics during the year. However, this practice also makes it more challenging to fill up a schedule

The percentage of teachers who disregard math circle flyers is often too high to make this means of contacting students viable for organizers.

and gives the math circle a bit of a guest lecture series flavor. (There are ways to counteract this effect, described in the section on retaining students.) Should a speaker come twice in a row? Fewer people may be willing to commit to two appearances, but the advantages gained include an increased sense of continuity between meetings, the ability for a speaker to build on material from the previous week, and a far easier time slating enough presentations for the year. The majority of math circle coordinators attempt to book

How many times should a speaker visit?

speakers for at least a couple of visits, particularly those who are known to be effective. At the other end of the spectrum, some directors arrange for instructors to conduct an entire block of math circles lasting a month or more in length. This approach allows students to become very familiar with the speaker, promotes regular attendance, and permits a more thorough treatment of a subject by the instructor. It also calls for a different model of math circle, in which speakers are paid for their time through the support of outside funding. The math circle might also choose to charge tuition to help guarantee the regular presence of excellent instructors; this issue will be addressed in more detail in an upcoming section on funding.

1.7 Getting the word out

Scheduling speakers may be the most challenging aspect of coordinating a math circle, but the task of successfully advertising the circle to students runs a close second. One of the most effective ways of notifying potential participants, of course, is to send a message directly to the students themselves. Various organizations that offer math or science programs for secondary schools might be willing to contact the students on their email lists on the circle's behalf. Naturally it is a good idea to write out a paragraph or two introducing the math circle which can be used within their message. This approach has the added advantage that the math circle is implicitly endorsed by an organization with which the students have (hopefully) already had a positive experience. Students heard about the Stanford Math Circle in this fashion, when the local

Contacting the students directly



Circle Snapshots



Name: The Math Circle

Location: various Boston area sites

Director: Robert and Ellen Kaplan

Email: kaplan@math.harvard.edu

Meeting time: various times

Web site: www.themathcircle.org

The following excerpt, slightly updated, is taken from the history given at The Math Circle web site:

Disturbed by the poor quality and low level of math education in the country, three of us (Bob and Ellen Kaplan, and our colleague Tomás Guillermo) began The Math Circle in September 1994. We rented space on Saturday mornings in a local church and word of mouth alone brought us twenty-nine students for that first (ten session) semester. Scholarships were offered when needed, and we almost covered our expenses.

The Saturday format we set up then continues still—but now on Sunday mornings to avoid soccer conflicts. The students are divided up into two groups (roughly by age) and now Gordon Ritter and Sam Lichtenstein teach a class from 9:15 to 10:00. After a fifteen minute break for juice, cookies and conversation, we change places and teach a different group from 10:15 to 11:00. In the last hour we all listen to an invited speaker or work on a joint problem.

In the second semester of our first year, Northeastern University (through Andrei Zelevinsky) offered us free—and much larger—quarters; and demand for a younger class led to a single session on Thursday afternoons, in a room offered us free by Harvard (thanks to Danny Goroff). Enrollment rose to thirty-four.

We began our second year at Northeastern and Harvard with thirty-eight students (some coming via the article about The Math Circle, which had just appeared in the AMS Notices), so we added a Harvard graduate student to our staff, setting a pattern which has allowed for our expansion to two hundred five students. The Sunday classes have, of necessity, stayed about the same size, but the weekday sessions for five to eleven year olds have proliferated: this year there are nine classes meeting at Harvard, on Tuesday and Wednesday late afternoons. A total of 121 students in all are registered this year.

We have been asked to give demonstrations of our approach, with the prospect of expanding, to public schools in Cambridge, Brookline and Taunton, and up and down the length of Great Britain. And perhaps most exciting, our book, *Out of the Labyrinth: Setting Mathematics Free* has just been published by Oxford University Press.

ARML coach and the Art of Problem Solving web site both sent out messages to students in the area less than two weeks before the initial meeting. Nearly fifty students (and a substantial number of parents) appeared the first weekend, forcing the event to move to a larger auditorium at the last minute.

Other methods for contacting students can be loosely classified according to the messenger. The most obvious choice of person to deliver a flyer or make an announcement would seem to be a math teacher, thereby ensuring that the news reached students at the appropriate grade level and geographic region. This strategy can work, and is in fact one of the best ways to reach “unenriched” or minority populations. However, results will vary dramatically depending on the amount of initiative displayed by the various teachers. As part of their publicity effort, the Mobile Math Circle mailed flyers to all teachers at nearby public high schools to announce their circle. They proceeded to discover that a disproportionate number of kids all came due to the efforts of one particularly enthusiastic math teacher. In the end, the percentage of teachers who disregard math circle flyers is often too high to make this means of contacting students viable for organizers.

*Teachers as
messengers*

As something of an extreme example, consider the cautionary tale of the University of Birmingham, whose math department took it upon themselves

*One teacher allowed
students participating
in the weekend
math circle to skip
the assignment due
Monday morning.*

to provide an uplifting mathematical experience for boys and girls attending the neighborhood city schools. A few organizers put together the entire math circle package, complete with a schedule of speakers, space to conduct Saturday morning events, even funding to provide transportation to and from local schools each weekend. Unfortunately, nobody thought to get the teachers on board. When a comprehensive mailing was

The missing link

sent out announcing the new math circle, the organizers received exactly one response, and the program folded before it began.

When a teacher can be found who is willing to steer students towards the math circle, it is important to impress upon them the type of student who would benefit most from a math circle. In particular, there is much lower correlation between grades in school and math circle appreciation than most teachers would expect. Students should attend a math circle out of a innate love for the subject: because they are fascinated by the clever patterns among Fibonacci numbers, because they can’t wait to see how number theory is used to enable secure transactions over the internet, or simply because they love to solve problems. Some of these kids are also A students, but some are not. To put it another way, some students earn top grades in math because they are academically competitive or

*Identifying
students who
would enjoy a
math circle*

Incentives for attendance

because they have become quite adroit at solving textbook exercises. These facets of classroom education are either downplayed or non-existent at a math circle, making the environment much less appealing to these types of students. As an illustration, one math circle encouraged teachers to offer extra credit to students who attended. Not surprisingly, this ploy attracted the wrong crowd, and was abandoned some time later. On the other hand, a teacher from another area allowed students attending the weekend math circle to skip the assignment due Monday morning. Since it was simpler for the academically minded kids to just polish off their homework questions, only those truly interested in the math circle consistently accepted her offer. To recapitulate, it is more important for a student to be interested in math than accomplished at math.

When it comes to penetrating bureaucracy, finding the right person makes all the difference.

Navigating bureaucracy

The return on advertising investment drops even further when attempting to reach students and teachers through school administrations. Experience has shown that most school systems simply ignore appeals to advertise math circles. Given the number of organizations that would benefit from contacting potential customers in this manner, one can hardly blame them. The exception to this rule occurs when a math circle is able to gain the support of an individual with connections to those who oversee or promote math enrichment within the school district. The San Francisco Math Circle illustrates a perfect example of this sort of phenomenon. They attempted to reach math teachers via central school offices for weeks to absolutely no avail. Fortunately, they were able to contact an independent consultant who helped set up meetings with math department chairs in the area. To the relief of the organizers, interest in this new math circle for students and teachers quickly sprang up, resulting in a much more substantial turnout than originally anticipated. When it comes to penetrating bureaucracy, finding the right person makes all the difference.

There are practically as many other creative means of disseminating information as there are math circles. Ideas that have been tried, or at least considered, include the following:

- Make an announcement in the weekly college newsletter sent out to all employees. This tactic can net a substantial number of faculty kids.
- Have hand-picked graduate students (or faculty members, better yet) give guest presentations at math clubs in local secondary schools, then distribute information about the math circle.
- Send a press release to local newspapers and TV or radio stations. One

math circle landed a spot on the morning news this way. Another received quite a few phone calls from interested parents after a widely-circulated paper ran a story on math circles.

- Find a parent who is willing to speak directly with teachers to advertise the math circle, since teachers can be intimidated by and hence dismissive of a university-based program.
- Encourage supportive parents to spread the word among families in their after school programs, churches, workplaces, or anywhere else.
- Set up a direct mailing of flyers to all residential addresses in the area announcing the circle. This approach requires some financial outlay, of course, but is hard to beat in terms of sheer impact.

The biggest hurdle to overcome is convincing students to attend once. After they have found time in their schedule, navigated parking lots, located the meeting room, and met a few of their mathematical peers, they will realize that math circles are truly wonderful events and will want to return regularly.

Chapter 2

Supporting a Math Circle

There are a number of issues central to the formation of any math circle. These include identifying the intended audience, selecting meeting times and dates, recruiting speakers, and publicizing the circle, to mention a few of the items covered in the previous chapter. However, several other components fundamental to math circle operation must also be taken into account. The topics discussed below are grouped together because they all function to support the operation of a math circle. These considerations include developing an administrative team, a web site, and a budget.

2.1 Delegation and collaboration

I once asked Richard Rusczyk, who coordinates the San Diego Math Circle, for any advice at all that he could offer a potential organizer. He immediately responded, “Enlist parent volunteers to bring in snacks each week. Then appoint one of them to oversee the whole process.” This may seem like a relatively small matter to top the list of possible pieces of advice, but it underscores a very important point: there are many more facets to a successful math circle than any single person should manage on their own. Handling the weekly task of reminding parents to bring snacks and then arranging for substitutes when the inevitable unforeseen conflict arises is a detail that can either wear down an organizer or provide the opportunity for a parent to assume a significant role within the life of the circle. Delegation is good—it transforms a potentially erosive task into a tool for enlarging the math circle community.

But delegation is also risky. An organizer may worry that it will only be a matter of time before someone forgets to bring the chips and there will be a hunger-induced riot to contend with. Some organizers may conclude that they

The most important piece of advice

*Delegate despite
potential snafus*

are best able to manage all the details involved with the operation of the circle. While it is quite possible that a volunteer will drop the ball at some point, this objection is not a valid argument against delegating the job. The flaw in this reasoning lies in the assumption that an organizer can control all factors necessary to guarantee a successful math circle. It helps to remember that the correct definition of success in this setting has to do with the extent to which the kids engage in and are captivated by mathematics; the degree to which all those attending feel that they are part of an exciting and worthwhile enterprise. With this perspective in mind, the possibility of going without chips for a week fades to insignificance.

*Tips for
successful
delegation*

To successfully delegate a task just ACT: Ask, Convey, Thank. So set out a highly visible form at the first meeting requesting parents to sign up for one of the dates listed on the sheet and to provide an email or phone number so that a reminder can be sent. Make sure that volunteers' duties are clearly delineated; for example, they should bring enough food and plates for twenty kids, set the food out at 2:45, then clean up at 3:15. Remind them of the golden rule of snack time: less mess is best. Instruct parents to avoid snacks like cake or popsicles, and if possible rely on a nearby water fountain instead of bringing beverages. Then ask the parent at the top of the list if they would be willing to be in charge of snack time by sending out a reminder each week and handling conflicts by arranging swaps with parents further down the list. The more clearly this person understands their responsibilities, the more smoothly the task will be handled. The goal is to anticipate and communicate possible eventualities so thoroughly that the organizer no longer needs to devote any attention to the matter. Finally, don't forget to express sincere appreciation at the end of the semester or year for the snack marshal's contribution to the math circle effort. A small gift is entirely appropriate and strengthens the bond with the community.

*Delegation is good—
it turns a potentially
erosive task into a
tool for enlarging the
math circle commu-
nity.*

*To delegate or
not to delegate*

There is a fine line between masterfully delegating chores and unloading legitimate responsibilities, determined primarily by the extent to which the task is essential to the operation of the math circle. Thus providing snacks, staffing the welcome desk, or even handling the math circle tutoring program all lend themselves well to delegation. The same is likely to be true of any extra components that the organizer chooses to incorporate into the circle, such as maintaining a lending library or holding an end of year party. (These ideas and more are outlined in the section on personalizing a math circle.) However, other tasks are best left in the hands of the coordinators. These responsibilities should

◦ ◦ ◦ ◦ ◦ ◦ ◦
◦

Circle Snapshots

◦ ◦ ◦ ◦



Name: San Diego Math Circle
Location: University of California, San Diego
Director: Richard Rusczyk
Email: rusczyk@artofproblemsolving.com
Meeting time: Saturdays 8:30–11:30am
Web site: math.ucsd.edu/resources/sd_math_circle/

◦
◦
◦
◦
◦
◦

Like many math circles, the San Diego Math Circle began in a somewhat unpredictable fashion. By the spring of 2003 Richard Rusczyk was already making a substantial impact on the high school mathematics culture through his online community at <http://www.artofproblemsolving.com>. As a result he was asked to discuss math contests and problem solving at a meeting of parents in the San Diego area. Among those attending was Professor Don MacLeod from the University of California, San Diego, who became eager to see Richard work with kids in the area. Thus the San Diego Math Circle began as a group of a dozen youngsters gathering weekly in the UCSD psychology department to work on interesting mathematics. The math department got wind of the event the following year and invited the group to relocate to their building, where they have been ever since.

Early in 2005 parents were encouraged to help recruit more participants and take on volunteer duties. The result has been staggering growth in the number of individuals who are part of the math circle—nearly 150 students attend each weekend. Students are divided into groups based on mathematical maturity level; roughly fifth to seventh grade, seventh to ninth grade, and high school. Richard and the other organizers invest a huge amount of effort to make each three-hour Saturday session an exciting mathematical experience for all students. Kids play fast paced problem-solving games, participate in engaging presentations, and consume snacks halfway through. Instructors include Art of Problem Solving staff, SDMC parents with a strong math background, UCSD graduate students, and occasional guest lecturers. There is a monthly math league event, a math olympiad, and a large scale picnic to conclude the year.

The largest key to the success of the San Diego Math Circle has been the involvement of the parents. One of these parents, Henry Lau, will become the new principal organizer beginning in January 2007. Others have been very involved in recruiting, teaching, and managing the students, among other tasks. Strong financial support has also played an important role. The Math Circle has a budget of roughly \$10,000 per year, which is used for prizes for contests, travel to the ARML competition, photocopying, guest speaker fees, and other expenses. These funds are managed by the Art of Problem Solving Foundation.

probably include scheduling speakers, obtaining funding, and taking charge of publicity, for example.

Still other duties fall squarely in the grey area between tasks that should be delegated and those that should not. Or perhaps an important matter related to running the circle falls outside the organizer's expertise. Establishing and maintaining a web site for the math circle is a perfect example of this sort of task. Not all of us have the time or skill to craft pages as comprehensive as the ones appearing at the Berkeley Math Circle web site! In cases such as this, there is an opportunity and need for close collaboration with other individuals who have a greater stake in the math circle than a volunteer. For instance, the director's husband created the Berkeley Math Circle site and an undergraduate on the work-study program is paid to maintain it.

Finding good help

It is not uncommon for a parent to provide the initial impetus for a math circle, in which case there is the potential for a fruitful partnership.

A group of ten middle school students meeting weekly on Tuesday nights to hear volunteer speakers benefits from the coherence of oversight offered by a single coordinator. On the other hand, a math circle serving a large metropolitan area offering material at three levels every Saturday morning requires several organizers with complementary responsibilities working closely together to function smoothly. Given the scope of the typical math circle, a steering committee of four or more usually becomes a bit top-heavy, although the San Jose Math Circle managed just fine with an initial group of five.

How many organizers are needed?

It is also not uncommon for a parent to provide the initial impetus for a math circle, in which case there is the potential for a fruitful partnership with a faculty member at a local college or university. The parent could work to contact schools, bring together students, provide snacks, and generally chaperone kids during the event. On the flip side, the faculty sponsor could arrange for speakers, reserve space, give presentations, and post information about the circle on the departmental web site. A collaboration of this sort has been quite successful at Washington University for more than five years. As a result a new mathematical opportunity exists for students in the area, and the university now has closer ties to the neighboring community as well.

Parent-faculty partnerships

2.2 Embracing technology

The internet has revolutionized our ability to gather and disseminate information. There's no reason not to put it to work for the friendly neighborhood math circle. Speakers searching for topics should be aware that many existing

math circles make handouts from past sessions available via the web. A minimal amount of further hunting will turn up a wide variety of mathematical enrichment ideas that can also serve as the basis for a math circle. On the flip side, most coordinators have ready access to server space via their institution or other sponsoring organization which could be used to host a math circle web site. It is worth the investment of time and energy to create a web site that is attractive and well-organized. The site can then function as a central clearinghouse for all items pertaining to the math circle for years to come. Indeed, a web site can become the face of the math circle to the outside world.

*Taking advantage
of the internet*

First and foremost, in order to be effective, a web site must be easily navigable. In other words, visitors to the site should be able to quickly find answers to a wide variety of questions with only a few mouse clicks. Spending a bit of time browsing through existing math circle web pages should give a sense of what sort of information to include at the site and how to arrange it. Displaying a list of the major sections of the site across the top or side of each page is usually quite helpful. As a simple litmus test, seat a friend who is not familiar with either the math circle or the web site at a computer and ask them a few basic questions, such as when the final meeting for the year will occur, where participants should park, what the tuition policy is, or how to contact the coordinator. Then watch their efforts to track down the answers.

*Designing a
navigable web
site*

It is also desirable to build an attractive site. To this end, consider setting up a consistent color scheme, displaying a common logo or institutional header across the top of each page, and employing common sense and good style when laying out text. FOR EXAMPLE, AVOID WRITING IN ALL CAPITALS. Large blocks of centered text are also a bad idea. Don't overload a page with too

*A web site can be-
come the face of the
math circle to the
outside world.*

much information, forcing the user to scroll through many paragraphs as they look for an item. Rather, split material into many smaller topics, each of which will fit on one screen. There is plenty more free advice on this subject available on the internet. (See <http://www.webstyleguide.com> for one comprehensive introduction to sound design.) Most coordinators

*Tips for sound
web design*

will want to enlist the aid of a colleague who has some experience with web page design. Many high school and college students are also quite adept. Be sure to provide this person with an overall plan for the organization of the site, thoughts regarding colors and motifs, and the text which will appear on the pages, known as 'copy.' Meet regularly during the construction of the site, since it is much easier to make midcourse corrections than sweeping revisions.

While every web site should be unique, there are some essential elements

that should appear on any site devoted to math circles. Here are a few ideas for content to include, listed roughly in order of importance.

Home page. Guests to the site will see this page first. Include a brief description and mission of the math circle, the level of exposition, the names and positions of the organizers, contact info, the time and location for getting together, details regarding tuition, important announcements, and similar pieces of information. Links to all other major sections of the web site should appear across the top or left side, and the entire home page should fit on a single standard screen, or at least come close.

Location. Assuming that a student has the time and interest, the biggest obstacle to getting involved is simply getting there. Take the time to explain in detail exactly where the circle meets: provide driving directions from different regions, mention landmarks to help parents navigate, supply tips on where to park, and indicate which door to take into the building. If applicable, also supply directions for taking public transportation. Include maps or links to maps if possible.

Schedule. Go into more detail on the times and dates of the meetings. Include names and affiliations of speakers, topics and titles of presentations, and perhaps also any handouts. Prominently display holidays or other dates on which the math circle will not be gathering.

FAQ page. It's not a bad idea to devote a portion of the web site to answering Frequently Asked Questions (FAQ). These could include "What is a math circle?", "Who can attend?", "Where is it located?", "How much does it cost?", and so on. Don't worry about repeating information located elsewhere on the site; just include links where appropriate leading to more detailed information.

Pictures. A few snapshots of actual students engaged in mathematics will go a long way towards making newcomers feel welcome. But be sure to obtain appropriate signatures first; there are guidelines regulating when it is permissible to post pictures of students at a web site.

Math problems. Include a few nice problems to illustrate the sorts of mathematics presented at the math circle. These could be displayed as graphics or written up in a PDF file. Include solutions separately, or invite students to submit solutions. More ambitiously, post a problem of the week and award prizes to solvers at the math circle meeting.

Links. A page of hyperlinks to related sites is practically de rigueur for any serious web site. Include links to other math circles, favorite math sites, other local mathematical events or organizations, sponsors, or anything else that seems appropriate.

And while on the subject of links, it is worth mentioning that savvy webmasters arrange for links pointing in the other direction as well. In other words, a web site is only as effective as it is accessible. Search engines help out a lot in this regard, but it is still worthwhile to contact the administrators who maintain sites for schools in the area to ask whether it would be possible to include a link to the math circle (along with a brief description) in a place where students are likely to come across it. The same would be true for other sites likely to be visited by students who enjoy math.

*Pointing students
to the web site*

Although the actual web site pages may reside on a server in a math department, it will often make sense to buy a domain name from an internet service that points to the real web pages. Their fee should be relatively minimal, on the order of ten dollars per year. Thus it would be far more memorable to invite participants to find information at www.abcmathcircle.org than to ask them to visit www.universityofabc/math/organizername/circle.html. Now that the internet has made obtaining data so potentially easy, users are more susceptible than ever to ignoring information that is not readily accessible.

*Creating a
memorable URL*

Coordinators inevitably find themselves in the position of needing to make urgent, last minute announcements, such as reminding students to bring their ruler and compass with them to tomorrow's meeting. Once again technology speeds to the rescue, providing a number of methods for keeping in touch with

The primary challenge is to convince students to actually join an email group; in practice less than half get around to it.

participants. The most straight-forward approach is to simply ask students to supply a current email address the first time they visit the math circle, and then use this registration information to maintain an email list. Almost all students have email accounts, check them regularly, and are willing to be contacted for a purpose as noble as a math circle. Just state clearly on the registration sheet that email addresses are never distributed to other organizations. Be sure to follow

*Keeping in touch
with students*

through on this promise and honor any requests to be taken off the list. One might also consider maintaining a similar email list for parents or teachers if the need arises.

Other methods for keeping in touch include user groups, bulletin boards, and other online forums. The former approach involves signing up with an internet service and then creating a group. Whenever messages are addressed to the

*Other means
for contact-
ing students*

group, everyone who has joined receives the email. In this way email addresses are kept confidential, even from the person who initiates the group. The primary challenge here is to convince students to actually join the group; in practice less than half get around to it. This approach requires a little more effort to set in place as well. The other option available to math circles is an online forum, which can provide a nice solution to the problem of having kids communicate with one another between meetings, collaborate on solving problems posed at the most recent session, and so on. The drawbacks remain the same, however: students are typically too busy during the rest of the week to think about the math circle, no matter how engaging it was at the time. So most organizers will opt for maintaining an email list and contacting students directly when the shortage of rulers and compasses becomes apparent.

2.3 Money matters

*Math circles are
worth it*

The concept of a math circle supported purely by the desire to enjoy math for its own sake has a certain appeal. This dream can be realized, as evidenced by the San Jose Math Circle, which thrived on a strictly volunteer basis for years. However, money spent on a modest year-end party, prizes for exceptional students, and books for the lending library does add up over the years. And while generous organizers can easily pay for these items out of pocket, a math circle is very much a joint enterprise between an institution and a community, so it is also appropriate for both sectors to contribute financially to the effort. Therefore the San Jose Math Circle recently began to accept donations to help cover expenses. Regardless of who foots the bill, any coordinator would agree that there are costs associated with running a math circle in the manner they envision.

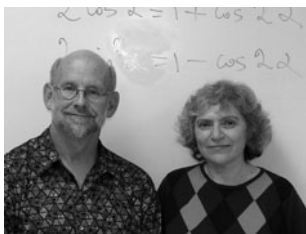
*Developing a budget
is an illuminating
process that helps to
shape and focus the
vision for a math
circle.*

*The budget is
central*

The precise amount of money that should be allocated and the corresponding source of this funding varies more dramatically from one math circle to the next than practically any other single attribute. But the planning process should always begin in the same way; namely, by creating a budget. The bottom line can range from a few dollars, to several thousand dollars, up to amounts exceeding one hundred grand. These budgets might cover occasional snacks, or supply honoraria for regular speakers and cover travel and lodging for several out of town visitors, or underwrite a large scale operation which provides stipends for teachers who receive college course credit for attending a teachers' math circle course while their students explore related topics in



Circle Snapshots



Name: The San Jose Math Circle

Location: San Jose State University

Director: Tatiana Shubin and Tom Davis

Email: shubin@math.sjsu.edu

Meeting time: Wednesdays 7:00–8:45pm

Web site: www.geometer.org/sjcircle/

Tom Davis received his Ph.D. in mathematics from Stanford, completed a post-doc there in electrical engineering, and became one of the founders of Silicon Graphics. He worked as Principal Scientist until his early retirement in 1998. Since then he has been devoting much of his time to teaching young people lively mathematics and creative thinking through problem solving.

Tatiana Shubin was born and grew up in the USSR. As an eighth grader she attended a specialized physics and mathematics boarding school in Siberia. Her experience interacting with working mathematicians kindled her passion for the subject and was very influential in her later choice to embark upon an academic career. Upon arriving in the United States, Tatiana earned her Ph.D. from UC Santa Barbara and joined the faculty of SJSU in 1985. At that time she started a math circle for local high school students, but it was discontinued after two years, mostly for lack of interest from students and colleagues alike.

Early in 1998 Tatiana and Tom heard a presentation by Zvezdelina Stankova on the math circles she had attended as a child in Bulgaria. She proposed that individuals in the San Francisco Bay Area launch circles in their communities as well. This idea resonated with both Tom and Tatiana, who felt that there now existed a critical mass of like-minded people to make the endeavor a success this time around. The San Jose Circle has been operating ever since.

Currently around twenty-five students attend each session; the large majority are in middle school. During the 2005-2006 academic year there were 23 meetings led by 15 different individuals. Among these were research mathematicians, such as Elwyn Berlekamp, Brian Conrey, Ioana Dumitriu, and Paul Zeitz; and experienced math circle leaders such as Joshua Zucker and Sam Vandervelde. With so many different leaders, the circle attendees see a great variety of mathematical styles. The San Jose Math Circle operates on a very small budget, funded primarily by donations. Nevertheless, there is a small library of books and math magazines that students can check out. The last meeting of every school year consists of the traditional pizza and puzzles party. Students and parents alike enjoy free food and a great variety of challenging puzzles.

a nearby classroom. Developing a budget is an illuminating process that helps to shape and focus the vision for a math circle. It is a document to which a coordinator will return time and again as practical decisions are made in the planning of the circle.

*Typical budget
line items*

The following list gives a sampling of the different categories that could constitute a math circle budget, along with considerations that will influence the relative amounts dedicated to each. The headings shown below are by no means comprehensive, but will at least provide general guidelines regarding some of the more common expenses. The last several paragraphs apply primarily to larger scale math circles that will reach out to teachers as well as students or that will be seeking philanthropic support.

Director compensation. This item may be omitted all together, come to a few thousand dollars, or equal the equivalent compensation for one university level class, on the order of \$20,000. Coordinators should choose an annual amount that allows them to sustain the activity over several years and, if possible, enough to free them of other responsibilities so that they can devote adequate time to the needs of the circle. For example, the coordinator of the Utah Math Circle is given release time in the form of one less course per year in order to devote attention to math circle administration. Coordinators tend to love their circles passionately, but are also subject to burnout. Adequate compensation permits them to invest sufficient time and energy in the circle while staying fresh and enthusiastic. Some organizers have started their circles with no compensation and then paid themselves in later years when resources were more plentiful. Even if an organizer feels strongly about donating their time, the budget should still accommodate incidental costs that inevitably accrue, such as manipulatives, overhead slides, or babysitting costs.

Speaker compensation. Some circles don't compensate their speakers, others offer them from \$100 to \$200 per visit. The expectation for compensation usually rises with the number of times the speaker will come. While many excellent speakers are happy to volunteer their time to lead math circle sessions, it is certainly good etiquette to show appreciation by some means—and an honorarium is one of the best. Speakers may have to pay for babysitting, transportation, or parking on top of giving of their time. It is also the case that speakers who know they are being paid for their efforts will spend more time polishing their presentation, are more willing to put together an effective problem set, and will think twice about canceling an engagement if a conflict arises. As worthwhile as math circles

are, it can be difficult for speakers to justify going the extra mile if they feel under appreciated.

Recruitment. Getting the word out to students and teachers might involve nothing more than a few well-directed email messages, perhaps with an announcement attached as a PDF file. For only a few dollars one can print simple flyers for posting at local schools. (With teachers' and administrators' blessings, of course.) With an amount in the range of \$2,000 a coordinator could mail glossy color brochures directly to all residences within strategic postal codes. Finally, around \$7,500 will retain the services of a consultant to interact with the local school district, unions, teachers, principals, and administration.

Snacks. As noted previously, it would be great to recruit a parent to coordinate volunteers who bring snacks each week. Remember to allow for a gift for the snack marshal, and also plan on food expenses for any social events that take place throughout the year. The Washington University Math Circle convinced a nearby pizza vendor to sponsor their circle and deliver a couple of pies towards the end of each meeting! However, if the organizer will be purchasing snacks each week, budget for it here.

Books and Prizes. If the math circle features a lending library then budget a modest amount to replace lost books and buy more copies of popular titles. Some coordinators might also wish to purchase an entire set of books to use as a supplement to their circles over the course of the year. Awarding small prizes, such as mathematical puzzles, for attendance or problem of the week solutions is another way to enliven a math circle and encourage participation at the same time. Some math circles host math olympiads and offer more substantial prizes for top scorers.

Supplies. There are any number of additional expenses that might crop up during the year. This line item might cover videotaping, Zome tools, or ingredients for a soap bubble demonstration, to name just a few of the possibilities.

Visitors. Inviting mathematicians from further afield to present guest presentations can add an element of excitement and anticipation to any math circle schedule. As mentioned previously, this speaker could also give a colloquium talk during the trip. Figure on approximately \$600 for travel, accommodations, and meals for each visitor. This amount might also be partially covered by the departmental colloquium budget.

Space rental. The vast majority of math circles use space donated by universities. Coordinators from outside the academic environment who seek the support of a math department in providing speakers should also request their cooperation in securing a space for holding the circle. If one must pay rent though, this is the time to budget for it.

Insurance. Some locations require liability insurance to protect the owners of the site from liability claims. Experience shows that insurance is rarely required—when mandatory it can usually be purchased for about \$500 per year.

Teacher compensation. Some circles that target “unenriched” kids, notably the San Francisco Math Circle, offer a stipend to teachers who bring students to take part in the math circle. This is a very expensive proposition, but one of interest to some funders. Teachers are also encouraged to come through the promise of free continuing education units which they receive by way of their participation in a simultaneous session tailored for educators. The San Francisco circle compensates teachers at the rate of \$100 per weekly event, and pays on the order of \$150 for each teacher for continuing education tuition. This line item for the San Francisco Math Circle exceeds \$30,000 but has been critical to its success.

Administrative overhead. It is never too early to take into consideration who will be managing math circle funds. An organizer applying for philanthropic support will want to work with a non-profit organization eligible to receive a gift or grant. (Assuming that the circle is not already a registered non-profit organization as set forth in IRS section 501(c)3.) The host institution, Mathematical Sciences Research Institute, or other local organization could serve as a fiscal agent. Ask them about administrative overhead—this is usually charged as a percentage of the grant.

Assessment. Those applying for philanthropic support will no doubt be stating objectives to achieve with the donors funds. Be careful committing to these objectives so that they motivate the donor, are realistically achieved, and can be measured without interference to the program. In one recent math circle grant, in a fit of enthusiasm to please the prospective funder, the application promised pre- and post-testing of the students, and projected significant improvement. No doubt there was improvement, but most students would have fled this friendly program at the prospect of tests. Fortunately, the promise was renegotiated with the funder.

2.4 Finding funding

And now for the moment of truth: where will the money come from? And less obviously, where will the money reside until it is spent? In very rough form, the answer to the first question is some combination of individuals, families, institutions, corporations, foundations, government agencies, or private donors.

In more or less corresponding fashion, the second answer is personal or business

Dreaming big and planning for a flourishing circle often precipitates greater success.

bank accounts, institutional coffers, or with non-profit organizations. For new math circles with negligible expenses the answer to both questions tends to be the checking accounts of one or more individuals, such as the coordinators, who have a clear interest in supporting the enterprise. This arrangement is a natural way to begin, especially for organizers who are unsure

whether their math circle will catch on among students. However, after a year or two of successful operation (and usually sooner than expected), the scope of the circle will have expanded sufficiently to incur more substantial costs, and the coordinators will want to take stock of how much money they are actually spending and where they might turn for additional financial support. In short, the time will have come to draw up and fund a budget.

Sources and repositories of funds

It is to the organizers' advantage to anticipate the implementation of outside funding and address this need in the initial planning phase of the math circle. Among other things, this will motivate a more expansive vision for the math circle. Dreaming big and planning for a flourishing circle often precipitates greater success. One of the most appealing methods for raising money is to simply ask families to make donations as they are able. So that this process runs smoothly, clearly communicate a suggested amount per year, invite more affluent parents to contribute above this amount, and welcome any family to forego a donation if they wish. In this way all interested students feel welcome to attend and a partnership naturally develops between parents and the coordinator to provide an excellent mathematical opportunity for their children. Since the level of funding will be somewhat unpredictable in general and non-existent at the outset, the coordinator may need to ante up a starting balance or solicit a zero-interest loan from another individual, to be returned once the circle has reached a more stable financial situation.

Donations from families

Fortunately, there is a third alternative to securing funds for the purpose of beginning a new math circle. In May 2006 the Mathematical Sciences Research Institute (MSRI) announced a mini-grant program that makes awards in the amount of \$1000 per year, renewable for up to three years in total. The funds

*The MSRI
mini-grant
program*

may be used for books, room rental fees, honoraria, transportation expenses, or administrative costs. The grantee, in turn, is requested to report on the activities of the math circle, host a visit by a representative from MSRI to one session of the math circle, and document at least one presentation to be posted at the national math circle web site and shared with other math circle organizers. To obtain more information about this mini-grant program or to submit an application, visit <http://www.msri.org/>, select “Events/Announcements” from the left sidebar, then click on “Minigrants for Math Circle Startup.”

*Managing
donations*

Any organization funded by donations from participants will obviously want their supporters to feel confident in their investment. For this reason an independent bank account, well documented, is important. This can pose a challenge for smaller organizations, such as a math circle. Ideally, the accounts could be managed by a non-profit organization which supports endeavors in mathematics, such as MSRI or The Art of Problem Solving Foundation. Donations then become tax deductible and parents would feel more secure in making substantial contributions. Receipts for parents should include a statement of the approximate value of the services they have received in return for their gifts. The host institution may also be able to handle accounts for a math circle, but expect significantly higher overhead rates. Alternatively, a parent could act as a third party, receiving and disbursing funds from a personal or corporate bank account. Regardless of the arrangement, the coordinator should be sure to keep a careful record of amounts received and spent which will be made available to all supporters on a regular basis, i.e. every three months.

Having invested financially in the math circle, parents are much more likely to encourage their children to attend on a regular basis.

Tuition model

If the parents of participating students tend to be well off financially, a coordinator should also consider charging tuition to raise funds. For example, the Boston Math Circle has found this solution to be the most effective recourse given their audience. The tuition model is straightforward and may be particularly attractive to circles that emphasize competition training. It has the added advantage that, having invested financially in the math circle, parents are much more likely to encourage their children to attend on a regular basis, thus combating attrition. Affluent parents are accustomed to paying fees for after-school musical, athletic, and other academic enrichment activities. There is a hidden price to pay, however, with tuition-funded circles. The relationship between the parents and the coordinator takes on a very different dynamic than with circles supported by donations. Expect more suggestions, complaints, and even demands from the parents in the circle with tuition funding. This may

not be productive if the parents are less than fully knowledgeable about the objectives and processes of the circle, and may not be worth the extra time and energy expended. And yet, parents are often the most ardent of supporters, since they tend to care deeply about their child's mathematical education.

Whenever a tuition model is chosen, it is recommended that the amount charged be set high enough to cover more than the cost of operating the circle. It is then possible to make available scholarship assistance to students who would

Don't hesitate to apply for a grant, even if it does not seem perfectly suited.

benefit greatly from but cannot afford to pay for the math circle. For the most part, there is no need to advertise specifics for scholarship qualifications; simply make clear that assistance is available and that students in need should contact the coordinator for more details. Except in cases of profound need coupled with

Scholarship considerations

interest in coming to the math circle, it is preferable to offer much reduced tuition as opposed to full scholarships. A similar effect can be achieved by combining a grant that covers a quarter to a third of the circle cost with tuition or contributions. This model is desirable because it provides a nice diversity of funding, reducing the circle's vulnerability to a financial setback. Grant makers, even as they are willing to support a math circle, will often want to know that the parents and "service recipients" are contributing a share. The process of securing grants is an entire subject in its own right, to which we now turn.

2.5 Granted

Multiple sources of funding help secure the long-term viability of a math circle, especially those with a broader scope and larger budget. Therefore it is advisable to conduct a thorough search for possible sources of support. Ask parents, guest speakers, and colleagues for likely prospects. The math department or host institution itself is a good place to start; they may have ideas for further contacts in addition to being able to provide partial funding for the circle. A good source for corporate and foundation prospects is the Foundation Center—visit a Foundation Center library in some major cities for free, or visit <http://foundationcenter.org/> to search for a fee. Also check to see whether large corporations in the area have an education foundation; many do. Seek out prospects that fund science or mathematics education. Call the person listed and discuss the project with a program officer, if possible. Make a note of the board of directors for good prospects, and share this list of names with parents and others interested in the circle. If someone knows a director, ask for their support in writing the proposal and in making initial contact.

Locating grants

○ ○ ○ ○ ○ ○ ○ ○

○

Circle Snapshots

○ ○ ○ ○



Name: The Mobile Math Circle ○

Location: University of Southern Alabama ○

Director: Vasiliy Prokhorov ○

Email: prokhorov@jaguar1.usouthal.edu ○

Meeting time: Mondays 7:00–8:30pm ○

Web site: www.southalabama.edu/mathstat/info/math_circle/ ○

Like many of the earliest math circles in the United States, the Mobile Math Circle was initiated by a mathematician who had been inspired by circles he attended while in high school in his country of origin. Vasiliy Prokhorov, along with Daniel Flath, decided to offer a pilot math circle during the 1999-2000 school year for students at the Alabama School of Math and Science (ASMS). The program was successful, so the following year the pair relocated to the University of Southern Alabama, their home institution, where the Mobile Math Circle continues to meet. In 2003 Natalya Prokhorova began teaching at ASMS and revived the former math circle, which now also meets on Monday evenings.

The style of their meetings closely resembles the types of circles that Vasiliy and Natalya enjoyed as students in Russia. Vasiliy describes a typical evening in this way.

The format for the sessions has always been the same. At 7:00pm the students are given a handout consisting of fifteen to twenty problems on a common topic. They are challenged to work on them one problem at a time. Students volunteer to present solutions or suggest ideas that might lead to solutions or they may ask questions. They may work together. They may come to the board to write out their thoughts for all to see. The students are not shy about thinking out loud, and they have become more and more active as the year has given them experience in the math circle. As the students work and comment, the teachers in the room may move about to talk with individual students. There are refreshments (cookies) available.

Anywhere from five to twenty-five students attend on a given Monday night. Like most math circles, the event is free and open to all interested students.

The Mobile Math Circle is supported by a yearly grant of \$5000 from the Alabama Space Consortium that is matched by the university. These funds enable the group to bring in regular guest speakers from around the country, host a math olympiad with prizes each winter, covers the cost of sending their top two students to the Colorado Math Olympiad, and helps to defray the usual administrative expenses associated with coordinating a math circle.

Most prospective funding organizations will provide clear guidelines regarding the sorts of projects they wish to fund and what elements should be included in a proposal. Don't hesitate to apply for a grant even if it does not seem perfectly suited, but remember to emphasize those aspects of the math circle which are most closely aligned with the foundation's purposes and goals. When drafting a proposal, give some details about the history and accomplishments of the institution hosting the circle. The organizers should describe their own background and particular interest in math and math circles. Outline the problem being addressed, such as bright kids who are not challenged in school or students who enjoy math but don't have a social group for support and growth. Segue into the proposed solution—a math circle—and the intended outcome. When predicting the positive impact that the math circle will have, remember to be persuasive without promising to conduct evaluations which will disrupt the event. A sample proposal is included in the appendices.

Overview of a grant proposal

Acknowledge contributions with a letter of thanks promptly—within 48 hours of receipt is best. It is important for many donors that the letter be dated and that the amount of the gift be stated in the letter. One is required to mention whether the donor has received any goods or services in exchange for

Begin work on a great grant report the week the grant is awarded, not the week the report is due.

the contribution. Since they usually have not, make a statement to the effect that “We acknowledge that you have received no goods or services in exchange for this gift.” (Basic recognition or gifts of thanks do not count as goods or services.) Become acquainted with the funder and their requirements or feelings concerning recognition. Many foundations and corporations

will be quite clear on this matter; expectations can range from the display of logos on printed materials and website in a specified font to strict anonymity. Others would rather cooperate with the one soliciting the grant to develop a plan. Some individual donors may be very difficult to read; err on the side of more recognition if forced to guess. Means for recognizing donors might include banners in the classroom with the sponsor's name, publicity in the media that mentions the sponsor (always with their permission), and mention on printed materials, especially handouts for students. With multiple donors, list them according to gift size. Thus a \$10,000 gift should be recognized approximately ten times more prominently than a \$1,000 gift. But also take into account the actual level of generosity: a \$1,000 gift from a donor for whom this amount represents a real stretch should receive relatively more recognition.

Acknowledging grant support

Individuals and organizations giving away money like to be kept appraised on how it is being used. The best time to begin work on a great grant report

The grant report

is the week the grant is awarded, not the week the report is due. If the funder does not make clear their expectations, ask them at the time the gift is made. Offer to provide an annual report and inquire whether they would like more frequent updates. Offer to provide specific data such as the number of meetings, their location, a list of speakers and topics, the size and composition of the average audience, and so on. Review the now funded proposal outlining what the math circle has promised to accomplish and set up a system for measuring the extent to which these goals are met. Pictures and anecdotal information—“A teacher reported that her student was talking about last night’s math circle session excitedly to all of her friends!”—are very useful. Be sure to submit the report when promised. A supplementary sample grant report accompanies this handbook.

2.6 Countdown to a math circle

Checklist for launching a new math circle

As a means of summarizing the issues discussed thus far and providing a sneak preview of others to come, here is a hypothetical to-do list for setting up a new math circle. For that matter, most items will still apply when preparing for subsequent years. No organizer will follow this schedule to the day, but it does provide a useful checklist when planning for the first meeting.

- ☐ **Six months.** Settle on an overall model for the math circle and identify the target audience. Partner with a local college or university, if applicable. Apply for grants or other sources of funding.
- ☐ **Three months.** Reserve space for conducting the math circle. Develop a web site for the math circle. Invite speakers.
- ☐ **Two months.** Set meeting times and dates for the year. Create a plan for publicizing the circle.
- ☐ **One month.** Advertise the circle. Continue to invite speakers.
- ☐ **Two weeks.** Continue to advertise the circle. Determine how to schedule the available meeting time.
- ☐ **One week.** Prepare forms and handouts. Plan the presentation for the first meeting.
- ☐ **Ongoing.** Invite speakers until the schedule is filled. Remind scheduled speakers of their upcoming presentations. Update the web site. Keep students informed of events. Involve parents in the life of the math circle.

Chapter 3

Sustaining a Math Circle

Having given thought to the issues related to the birth of a new math circle, the time has come to consider its nurture and development. One of the foremost challenges that an organizer faces during the course of a year is to reach and maintain a comfortable size. Some drop-off in attendance during the first few weeks is to be expected, but ideally the number of students will not fluctuate much beyond that. The organizer will also address other matters as the year progresses, such as how to propel especially motivated participants further along in mathematics, or how to handle the weekly maintenance of the circle most efficiently. However, the latter topics hinge upon sustaining the level of interest among students that brought them to the circle in the first place, so we begin with this issue.

*Reaching a
suitable size*

3.1 Retaining students

Having expended considerable time and energy to create an opportunity for students to broaden their mathematical horizons, it is enormously affirming to see a solid group of kids taking advantage of this opportunity week after week. Assuming that there is a sufficiently large population of students in the area who are genuinely interested in what a math circle has to offer, it is not hard to establish a core group of faithful participants. There are a few common sense principles to observe which will help to ensure that attrition never becomes an issue. (It can be one of the more discouraging issues—to be avoided if possible.) A few of these principles have already been touched upon, such as choosing an appropriate meeting time based on the intended audience or asking families to invest in the circle financially by encourage donations or charging a nominal tuition. The remaining suggestions are all related to activities that occur at the

*Maintaining a
suitable size*

math circle itself as opposed to decisions made during the planning phase.

*Conjuring up
community*

In the absence of extra incentives, students will attend an event such as a math circle if they (or their parents) feel that they are part of a worthwhile activity and that their presence is significant. The quality of the mathematical content and its presentation is the subject of the second part of this handbook; for now let us consider ways in which students can be made to feel that they are meaningfully connected to the math circle. The task is to turn a collection of students of various ages from a number of schools with differing mathematical backgrounds into a cohesive community. This process is more of an art than a science; as such, it is not necessarily the forte of every mathematician. However, the following ideas can be implemented by any organizer who wishes to build a thriving math circle.

The more a participant feels known by the coordinator and by their peers, the more likely they are to attend regularly.

*Establishing
continuity*

One of the most powerful means for accomplishing this goal is to establish a thread of continuity from one meeting to the next. Adhering to a uniform format, location, and meeting time is essential. If possible, the coordinator should attend every meeting in order to welcome students, make announcements, and introduce speakers. The regular presence of the person in charge helps to make the circle seem more familiar and accessible to students. Better yet, arrange to have speakers come for two or more weeks in a row, thereby halving the number of new faces that students must contend with and allowing students and instructors to become acquainted. (Recall that one should be prepared to offer non-trivial honoraria in this case.) Another effective strategy is to involve students in endeavors that span the course of several weeks or an entire semester. For example, at the first meeting students could be given a “Math Circle Challenge” consisting of a sheet of problems of varying levels of difficulty. The first ten minutes of each session could be devoted to presenting solutions, with the promise that a prize for the entire group would be awarded on the final meeting of the semester commensurate with their collective progress.

*Helping students
to feel plugged in*

It would be hard to overstate the impact that is made by an organizer who takes the time to greet students by name as they arrive, asks them if they have come across any interesting mathematics lately, or lends them an interesting puzzle to play with. The more that a participant feels known by the coordinator and by their peers, the more likely they are to attend regularly. Name tags for everyone at the first several meetings might help, although this approach is more likely to be effective at the middle school level. Use the initial moments of the meeting to run activities that help students connect with one another.

For instance, one could instruct everyone to write down a secret positive integer from 1 to 20, with the caveat that they should select a number which they think that nobody else has chosen. (Adjust the range as necessary based on the size of the group.) Those who perform the task successfully win chocolates, those who don't will have at least created a small bond with those other members of the circle who read their minds and spoiled their chance for a treat. By the way, this simple game quickly leads to some interesting mathematical questions in probability and expected value.

An ice-breaker

Along the same lines, incorporate plenty of time for mathematical and social interaction. Encourage speakers to set aside time for kids to work together on problems. Many topics can be motivated by simple games or numerical investigations upon which students can embark in pairs. Have students present ideas and solutions on the board, thus giving them a stake in the proceedings. Schedule time during or after the session for snacks and chatting. Plan a holiday party (with a mathematical theme, of course) or an ice-cream social for the second half of the final meeting of each term. The

*Incorporating
some social
interaction*

Students often need an extra nudge to attend an event they would sincerely enjoy and benefit from once they arrived.

group could also arrange to attend an event of mathematical significance together. While the social aspect of a math circle should certainly not become its focal point, a healthy amount of social interaction is important to its well-being.

Incorporating special events into the calendar also serves as an effective antidote to attrition. Math circles in the San Francisco Bay Area all turn their attention

to promoting the Bay Area Math Olympiad (BAMO) in late February and helping students to prepare for this intense proof-oriented competition. Everyone subsequently attends a lavish awards ceremony hosted by MSRI during the second weekend of March to hear a fabulous guest lecture, honor the high scoring students, and enjoy a nice lunch. Most circles have various special events spaced throughout the year, ranging from parties to special guest speakers. These events break up the routine of weekly presentations, provide a source of focus and anticipation about which to rally, and add momentum to the year, all of which contribute to healthy attendance.

*Special events
are beneficial*

Doubtless an organizer will want to drum up enthusiasm for an upcoming special event, which leads naturally to yet another means of retaining students: staying in regular touch. Several methods of communicating with participants have already been advanced—choose the method that best fits the group, then put it to good use. (But not overuse.) It is helpful for a student who has missed a few sessions to receive an email announcement regarding the need to bring a

*Keeping in good
contact*

Give them a reason to return

ruler and compass to the next gathering or reminding them of the special guest lecture for AMC-12 contest preparation. Students inevitably have legitimate reasons for missing several sessions in a row, at which point a potential barrier to renewing regular attendance materializes. The goal is to counteract this hurdle by offering students good reasons to come to at least one more math circle session. High school students in particular are bombarded with many demands on their time, and often need an extra nudge to attend an event that they would sincerely enjoy and benefit from once they arrived.

3.2 The Fountain of Youth

Legend has it that Ponce de León discovered Florida while searching for the Fountain of Youth. While this tale is almost certainly not true, the belief in an elixir that will restore youthful vitality or even reverse the aging process continues to persist. And while most math circle coordinators would welcome a rejuvenation of mathematical energy among its participants, not everyone is interested in decreasing their actual age. Yet this is precisely what can and does happen in many instances, at least to the average age of the students attending.

High school math circles look good on paper

The groups most vulnerable to this Fountain of Youth effect are those aimed at high school students. As noted previously, this age group is attractive to coordinators in a college math department for several reasons. For instance, students are closer in age and mathematical maturity to those whom the coordinator are accustomed to teaching. In addition, the list of potential mathematical topics which could be presented to such a group includes many more fascinating ideas than would be available to a middle school audience. Why then are math circles devoted solely to high school students so scarce?

Why do middle school students appear?

The Stanford Math Circle provides a typical illustration of how the Fountain of Youth effect operates. Prior to its first meeting the circle was advertised only to high school students and teachers. Topics selected during the fall (including Gaussian integers, geometry of complex numbers, and topology) were tailored for an older audience. However, word quickly spread among parents looking for mathematical enrichment for their middle school student with a scientific inclination and light homework load. A tally taken of all students who attended the Stanford Math Circle at any point during the winter session revealed that approximately a quarter of them were in eighth grade or below. Several factors help to guarantee their presence: middle school students typically have

Speakers may begin to unwittingly lower their level of exposition in response to young faces and elementary questions.



Circle Snapshots



Name: The Stanford Math Circle
Location: Stanford University, Stanford, CA
Director: Sam Vandervelde
Email: samv@math.stanford.edu
Meeting time: Sundays 2:00–4:00pm
Web site: www.stanfordmathcircle.org



The Stanford Math Circle came about because the coordinator-to-be happened to be in the right place at the right time. Shortly after completing his graduate studies, Sam Vandervelde found himself in Palo Alto when his wife began a post-doctoral fellowship at Stanford University. Aside from two boys and an apartment to look after he had relatively few other responsibilities once the move was complete. (This quickly changed.) Since he had always enjoyed presenting exciting high school mathematics and also had an informal affiliation with Stanford, he jumped at the suggestion made by a friend during the summer of 2005 to begin a math circle. In approximately a month's time supporters sent email out to a large number of local high school math aficionados, Professor Ravi Vakil became the official faculty sponsor and arranged for space in which to meet, and Sam cobbled together a web site, a few forms, and an introductory set of presentations. Over forty students attended the first meeting and the math circle is still going strong, now in its second year.

Between twenty and thirty students make their way to a large classroom in building 380 on a typical Sunday afternoon. Sam begins each two hour session with a brief warm-up (a nice problem, puzzle, or idea) before making announcements and introducing the speaker for the day. The rest of the time is devoted to the pursuit of great mathematics, with a short break in the middle for snacks, which are supplied by parent volunteers. The fall ice-cream social and winter pizza party are quickly becoming annual traditions, as are a day devoted to an action packed competition of some sort, an AIME practice session, and a trip to the Bay Area Mathematical Olympiad awards ceremony.

The Stanford Math Circle operates on a yearly budget of approximately \$1200, which allows for modest honorariums for speakers, food for the social events, and other incidental expenses. The budget is funded entirely by parental donations, which are managed by the Art of Problem Solving Foundation. Registration data indicates that the majority of students are in grades eight through eleven, with a few seniors and younger students as well. An email list is maintained and students are kept abreast of news via occasional email announcements.

fewer responsibilities and hence more free time to devote to such activities, they are often bright enough to understand enough of what is presented to be truly captivated by the material, and they are still substantially influenced by their parents' decisions regarding their activities and reliant upon them for transportation. Indeed, some of the most consistent participants that winter were also the youngest.

The effect of a middle school contingent

Once a significant proportion of the audience falls into the under-fourteen age range, at least two things are prone to happen. First, high school students may start to feel self-conscious about hanging out with younger kids. Expressing a public interest in math is uncool enough at many schools; doing math with middle school students might be too much. It doesn't help when younger siblings tag along, and the situation becomes even more uncomfortable when there is a little hotshot who is actually as quick at answering questions and coming up with bright ideas as the older students. At a more significant level, speakers may also start to unwittingly lower their level of exposition in response to the young faces and elementary questions. Since the younger students are often the most enthusiastic to raise their hands, this can be an issue even for speakers who are presenting more advanced material. Math circle coordinators are especially susceptible to watering down material since they are the most familiar with the group and most eager to boost attendance by ensuring that the mathematics is accessible to everyone.

Middle school math circle students don't necessarily become high school math circle participants.

Responding to middle schoolers

There are several remedies for this situation which help middle school students feel welcome without discouraging the older students. One of the simplest is to encourage parents or other adults to attend, thus artificially raising the average age. Once the math circle setting feels less like a school classroom and more like an event of interest to the general mathematical public, the issue of age range will become less prominent. It is also important to maintain a steady diet of advanced topics (but not too advanced!), even at the risk of leaving behind some of the middle school students. They are not as easy to intimidate as most coordinators believe, and the ones who are willing to be stretched will benefit a great deal from the presentations regardless. It is good policy to set clear expectations during the first meeting with respect to how much of the material students are likely to absorb. At the Berkeley Math Circle newcomers are advised that to follow even a third of any given talk is a significant achievement.

Occasionally students who attend for several years may reach the point where they feel they have mastered the math circle curriculum, such as it may be. Thus a Stanford Math Circle parent raised the concern, in a very diplomatic and con-

structive manner, that material was being pitched too low for his olympiad caliber daughter. The discussion was productive and led to the conclusion that, rather than raising the level of exposition and leaving the majority of the remaining students behind, it was time for this particular girl to move on to more advanced offerings. (These might include top flight summer programs, regional or national olympiads, or college courses, for instance.) In a sense this scenario is ultimately the goal of any math circle: to attract students interested in mathematics and develop them into such proficient and experienced problem-solvers that they are ready for greater challenges.

Yet another recourse for math circles with a sufficiently large audience involves splitting the group into two or more tracks based on some combination of age and mathematical experience. This has proven to be a successful approach in Boston, Berkeley, San Diego, and San Francisco, for example. Many aspects of coordinating such a circle, such as filling the speaking schedule, become twice as taxing as before, so be sure to muster additional staffing support. Ideally a secondary coordinator could take over the middle school group, as occurred at the Stanford Math Circle.

It is worth mentioning that the Fountain of Youth effect doesn't automatically resolve itself over time as students grow older. In other words, middle school math circle participants don't necessarily become high school math circle participants. A math circle might be the only outlet for younger students

Those on a mission to spark interest in math will want to focus their efforts on middle schoolers.

interested in science; as they grow older more opportunities become available and the same students may opt for the robotics team or become engrossed in developing their school's web site. Similarly, as students become increasingly independent of their parents they may decide that they are actually more interested in the a cappella group or drama club than in math. Yet another factor at work to discourage high school attendance is the fact that participation in a math circle is not perceived to carry nearly as much weight on a college resume as does recognition of achievement in competition, even though these prizes are often much less meaningful. Some math circles have incorporated olympiads or other types of productive contests into their circles, not only to give kids an arena in which to concentrate on solving challenging problems, but also to be able to make awards to their top students. Another organizer is planning to act as a mentor to older students interested in developing math projects for science fairs or papers for talent searches. Students often need direction when developing fruitful problems to tackle. They would then work independently to discover and present solutions.

Matriculation

Divide and conquer

Factors that influence high school attendance

Why to consider math olympiads

*Mission to
middle schools*

To be fair, there are many excellent reasons to arrange a math circle for younger students, say in grades five through eight. This window of time is arguably the most effective one for spreading the mathematical gospel, since many students make up their minds in terms of whether they “like math” or are “good at math” during these years. In fact, individuals on a mission to spark interest in the subject will probably want to focus their efforts on a middle school audience. The point is that coordinators interested in reaching older kids must be proactive in finding ways to keep a math circle targeted for high school students attractive to that age group.

3.3 MyMathCircle

*Personalizing a
math circle*

The discussion so far has centered primarily about the fundamentals of math circle operation, such as scheduling, publicity, and funding. But now we interrupt the relentless stream of logistics to consider some of the less essential facets of a math circle. There are any number of ways that coordinators can add their own personal touch, ranging from a clever logo to a lending library. These extras contribute far more to a circle than the length of this section might suggest. They are likely to represent the coordinator’s favorite aspects of the undertaking; these are the elements of the math circle which make organizing it a delight rather than a burden. Below is a cross-section of ideas taken from various math circles.

- The Berkeley Math Circle holds a monthly “Winner’s Handicap” contest, run by a graduate student, in which participants are given four weeks to compose solutions to a set of five problems. The coordinators evaluate all the papers and award prizes to the top students. Then in subsequent months the stronger students are handicapped based on their scores on previous contests to help level the playing field and encourage the less experienced students to continue to work diligently on the problems. The Berkeley Math Circle has also created a nice logo involving a verdant tree featuring the digits of pi superimposed across its leaves with the Golden Gate Bridge in the background. This graphic appears at the top of their web site and on other materials associated with the math circle.
- The St. Mark’s Institute of Math puts out a monthly newsletter featuring an introductory mathematical morsel, an interesting exposition, and a variety of questions (even open problems!), all centered about a common theme. The newsletter is published in hardcopy form only; the purpose is to populate the coffee tables in teachers’ lounges across the greater Boston

area with these brightly colored sheets of paper in the hopes that they will engage many more people than appear on the mailing list. The newsletter is currently mailed out to approximately 600 teachers and students.

- The Washington University Math Circle likes to send all participants home each week with an item related to the math that has just transpired. This might be a worksheet of problems for further investigation, or it could be some other type of souvenir, such as a set of linked Mbius strips resulting from an investigation in topology.
- The Stanford Math Circle presents warm-ups during the first ten minutes of each meeting. These might range from digit puzzles (“How many ways are there to make a perfect square with exactly four 1’s?”) to more meaty mathematical morsels (“How is it possible to fill all of three-dimensional space with non-overlapping circles?”) In this manner the meeting can begin promptly while still allowing leeway for latecomers to hear the entire presentation.
- The San Jose Math Circle wheels in a large collection of math texts and problem books each week written for secondary students. Students can browse titles during any free moments and then check out books on the honor system. The coordinator also regularly distributes other items of interest, such as past issues of *Math Horizons*, the MAA journal produced for undergraduates.
- The Mobile Math Circle holds an internal olympiad in early March as a culmination of the many problem solving sessions they conduct throughout the year. The organizers recognize all the students who made significant progress, presents book awards to a couple of high scoring participants, and rewards their top two students with an all expenses paid trip to the Colorado Math Olympiad event later that spring.
- The San Francisco Math Circle hosts a year-end picnic. Families of all the participating students are invited to gather at Golden Gate Park to spend an afternoon together. Besides engaging in the expected outdoor activities, kids also take part in an informal ceremony recognizing outstanding participants.
- The Sudbury Math Circle, which is based at a private school in Sudbury, Ontario, devotes an entire school day to holding their annual “Math Challenge Day” for their own students and any others who are able to attend. The students take a math contest in the morning, followed by a math

circle event from an invited speaker. A local pizza sponsor provides lunch, after which the students have recess and hear a second presentation by the speaker. Parents arrive, prizes are awarded, a reception is consumed, and everyone returns home exhausted but elated.

- The San Diego Math Circle runs math games as part of each meeting. Students attend a presentation during one half of the Saturday meeting and take part in an exciting, high energy team event during the other half. There is a snack break separating the two halves of the morning. This format permits speakers to work with each age group. The San Diego kids also sport cool T-shirts.

3.4 Paperwork

This section will be kept short and sweet. Administrative tasks may not top every coordinator's list of favorite activities, but there is no substitute for being prepared ahead of time with useful forms to ensure that a math circle runs smoothly. Some common aspects of a circle's operation which may require preparing a form are listed below. Samples of many of these appear in the corresponding appendix to provide a guide for preparation. The only one that is really essential for a moderately sized, informal math circle is the registration form; include others as necessary.

Organizers might choose to group information by student, so that each person attending receives a single sheet of paper which contains details about the math circle, lists ways in which families can get involved, provides instructions for donations or tuition, requests information about the participant, and perhaps includes waivers and space for signatures. Alternatively, the organizer could collect information by topic, so that one sheet of paper contains all registration information, another is used for attendance, a third handles volunteer sign-up, and so on. Each approach has its advantages; for example, the latter option makes more sense if a helper will be creating an email list, since all the necessary information will already be together in one place. Choose whichever option seems most appropriate.

It is good practice to get in touch with speakers to fill them in on what to expect on the big day.

Be prepared

Options for disseminating information

- **Registration.** It is extremely useful to track information on all individuals who attend the math circle. At the very least it is helpful to know each student's name, grade level, school name, and email address. Declare unequivocally at the top of the form that students' personal contact

information will not be distributed to any other individuals or organizations, then abide by this policy. In the event that someone does have a legitimate reason to reach a student, offer to contact that student on their behalf so that the student has a choice of whether or not to respond.

- **Attendance.** Recording who is present serves many purposes. Knowing total attendance each week and the change of this quantity over time provides a rough gauge of student interest and can be important to include in a report of the circle to those funding it. It is also helpful to know the number of students at each grade level who are participating on a regular basis. In addition, one can present awards for perfect (or nearly so) attendance.
- **Welcome.** In lieu of meeting with parents, an organizer might prepare a letter for them to pick up the first time they attend. This letter could begin by expressing appreciation for transporting kids and describing the wonderful speakers lined up for the year. It could then go on to provide an overview of the circle's operation, discuss ways that parents could volunteer or otherwise become involved with the circle, and explain how donations or tuition works, if applicable.
- **Volunteering.** One means of obtaining volunteer support for chores such as providing snacks each week is to set out a form asking parents to sign up for various dates. Leave space for volunteers to provide their name, phone, and email address for each date or task. Another creative approach (inspired by a kindergarten teacher) consists of writing "wish list" items on post-it notes so that parents can sign up for providing items or performing duties for the circle by simply removing a note and taking it with them. These requests might include one prize for the monthly contest, signs for directing new students to the room, email list maintenance, or anything else that a coordinator needs on a regular basis.
- **Consent forms.** Some math circles will need students to sign waivers in order to abide by university policies for use of rooms or for insurance purposes. These may include agreements not to sue the university in case of injury or to abide by all math circle policies. It is also necessary to obtain consent from parents of minors to post their pictures at the web site or include them in video clips.

Besides keeping attendance and updating email lists, there are a few other weekly administrative tasks that bear brief mention. For example, it is good

Weekly tasks

practice to get in touch with upcoming speakers to remind them of their presentations, arrange for any special needs such as a laptop projector, request a sheet of additional problems, and generally fill them in on what to expect on the big day. In the same manner, some organizers like to send a description of upcoming meetings and make announcements each week to all students on the email list. Other weekly tasks might involve updating the web site with the latest handouts or assembling questions for warm-ups or math games. No doubt each organizer will quickly develop their own weekly routine.

3.5 How am I doing?

Why to find out

Obtaining feedback on a math circle is a fairly natural impulse for a coordinator. This person has initiated the circle with a particular purpose in mind, presumably out of a desire to provide an exciting mathematical opportunity for secondary students. So it stands to reason that the coordinator will be interested in finding out how successfully this purpose is being achieved. Although the director probably has a fair grasp of the overall effectiveness of the circle, it is very easy for this person to get caught up in organizational details and miss important trends or shortcomings. Therefore it is important to pause periodically to solicit friendly but frank feedback, or otherwise objectively gauge the health of the math circle.

Getting inside information

Praises and gripes are freshest in students' minds immediately following the presentation, so they are most likely to express their feelings to their friends outside the building or to their parents during the trip home. Hence parents can be an excellent source of information. Make a deliberate effort to say hello occasionally to parents standing alone or (better yet) chatting with other parents. Inquire whether their kids are enjoying the math circle and ask directly if the parents would recommend any changes based on what they hear from their children. This form of question avoids the dilemma for parents of revealing students' criticisms verbatim, but keeps the focus on the kids as opposed to opening the floor to suggestions for steering the circle towards the parents' vision of what a math circle ought to provide for their college-bound son or daughter.

The other method of discovering students' reaction to the math circle is to ask them by way of an anonymous survey. Limit the questions to those items that are most important. Most students only have the patience to answer one or two open-ended questions thoughtfully, so make them

Don't wait until the end of the year to take stock; the time is ripe at any point five or six meetings into the schedule.

count. The survey might cover aspects of the circle such as the level of the material (too easy or hard?), the topics covered (routine or fascinating?), the style of presentation (droning or engaging?), what the students liked, and what they would change. For example, after conducting a survey the coordinators of the Washington University Math Circle discovered that students were regularly lost during talks, and were able to adjust the level accordingly. Don't wait until the end of the year to take stock; the time is ripe at any point five or six meetings into the schedule. As motivation, make a completed survey the students' ticket for snack or pizza that day. A sample form is shown in the appendices.

*Scintillating
surveys*

Coordinators who will be making reports for individual or corporate donors supporting the circle need to be even more deliberate about gathering feedback.

*There is no substitute
for testimonials when
it comes to securing
support.*

In addition to the above approaches it is important to settle on an objective yardstick for measuring the success of the circle before the academic year gets underway. Attendance figures and breakdown of audience composition by gender and race could all be important, depending on the stated goals of the original funding

*Planning for
grant reports*

proposal. Be sure to write down positive student and parental testimonials as soon as possible after hearing them; they fade from memory all too quickly. Take pictures of students actively engaged in mathematics. Record any honors garnered by students at the math circle as they come to light; this item might also be included on a follow-up year end survey. The sample grant report in the appendices can also suggest further data to collect.

3.6 Paging all parents

Several math circles, including those in San Diego and Berkeley, host a yearly meeting especially for parents of students attending the circle. This event is usually held at the same time as the regularly scheduled math circle at some point during the autumn; perhaps during October or early November. Such a meeting is a sound idea for any math circle which relies on parents in a significant way, financially or otherwise. This sort of gathering is particularly well-suited for circles in which regular contact with parents is minimal or non-existent; for example, if parents are only able to drop students off at the door of the building due to parking restrictions.

*Why to schedule
parent meetings*

There are many good reasons to make an annual tradition out of meeting with the adults associated with a math circle. Just as with Back-to-School nights, there will probably be no better chance to describe in glowing terms the positive mathematical impact a math circle can have on students and to

outline exactly how parent's contributions are translated into bringing about this experience. In the same breath, one could call for volunteers to help with the math circle. Finally, parents would appreciate the chance to ask questions and learn first-hand what their children have been doing all those weeks.

A sample parent meeting agenda

In light of these purposes, a sample agenda for the fall parent meeting might look something like the following. To begin, acknowledge all the people and organizations who have made the math circle possible. This would be a good time to invite a representative to say a few words on behalf of a major donor, if applicable. One might also recognize the host institution, regular instructors, and volunteers who have helped out in the past. The next logical step is to give an idea of what transpires in a typical math circle. The most effective means would be to conduct a miniature ten-minute presentation on a suitably enticing yet elementary topic. Alternatively (or additionally) a student or two could talk about what usually takes place and why they enjoy participating. There is no substitute for testimonials when it comes to securing support.

When the San Diego Math Circle enlisted the aid of parents in order to increase attendance, the size of the group exploded from less than twenty to around one hundred fifty students!

Following up

Once parents have a sense of what occurs and why it is worthwhile, take a few minutes to explain in clear and specific terms how parents can support the math circle. These might include details on how to make donations, where to sign up for bringing snacks, what transportation needs exist, or whether parents could help to advertise the circle at local schools to build the program. Finally, open the floor for questions and comments. Ask parents if they have any observations or suggestions to make based on feedback they have obtained from their children. Aside from fielding questions, the end of the meeting is also an appropriate time for alerting parents to other resources for their children, such as math olympiads or summer programs that the organizers endorse.

The power of parental support

When rallied effectively, parents can be a powerful force for advancing a math circle. When the San Diego Math Circle enlisted the aid of their parents in order to increase attendance, the size of the group exploded from less than twenty to around one hundred fifty students! Berkeley Math Circle parents organized a subsequent meeting of their own initiative in which they hatched a plan for a Christmas party where students would create mathematically-themed tree ornaments and enjoy cookies and punch. The moral of this story is not to underestimate the resources available to math circle organizers in the form of parental support, and to seriously consider holding a meeting to mobilize this support each fall.

◦ ◦ ◦ ◦ ◦ ◦ ◦
◦

Circle Snapshots



Name: The Berkeley Math Circle
Location: University of California, Berkeley
Director: Zvezdelina Stankova
Email: stankova@math.berkeley.edu
Meeting time: Tuesdays 6:00–8:00pm
Web site: mathcircle.berkeley.edu

◦ ◦ ◦ ◦



◦
◦
◦
◦
◦
◦

Zvezdelina Stankova is a Bulgarian math circle success story. As a fifth grader she was originally drawn to her middle school math circle when she discovered that her peers were learning how to approach tough problems that she was not able to complete. She rapidly became very adept at problem solving and ultimately represented Bulgaria at the 1987 and 1988 International Mathematical Olympiads held in Cuba and Australia.

College and graduate school brought Zvezda to the United States, but her involvement with math circles started only after her arrival in Berkeley, when she began her post-doctoral work at MSRI in 1997. During her first year she helped to advertise the concept of math circles to middle and high school teachers in the area, anticipating that they would pick up on the idea and launch circles within their own schools. When it became clear that most teachers did not have the requisite background (either with the extra-curricular mathematics or with the tradition of math circles), she proposed to coordinate a temporary math circle at the University of California, Berkeley which would serve as a model for teachers. This “temporary” math circle is now entering its ninth year.

Several years after commencing, it became clear that a select group of highly talented and motivated kids would benefit from more advanced topics and a greater emphasis on problem solving. So for a while two groups met concurrently. One of the highlights of being the coordinator occurred when two of her math circle students represented the United States at the IMO; in a sense Zvezda had come full circle. Not long after the demands of offering an advanced track became too great and the group transitioned to its current state in which it is comprised mainly of seventh through eleventh graders.

The Berkeley Math Circle began with a modest \$1000 budget to pay for prizes for their monthly math contest and cover administrative costs. It then ran purely on volunteer enthusiasm for five years before MSRI located outside donors to help expand the scope of the program. As a result Zvezda is now co-director of the Bay Area Math Olympiad in addition to being the director of the Berkeley Math Circle. A more thorough description of these programs and the impact they are having in the Bay Area can be found in the supplemental grant report accompanying this handbook.

3.7 Going the distance

*Keeping in mind
one's original
motivation*

At an opportune moment, pause to reflect on what aspect of coordinating a math circle is most appealing. Whether this be a love of teaching, a desire to reach underserved kids, or an interest in bringing students to campus, it is important for coordinators to identify the elements of the math circle that they find to be the most motivating and inspiring and then ensure that these elements remain at the center of their involvement with the math circle. Circumstances inevitably change, and it is surprising how easily coordinators can find themselves spending the majority of their time dealing with tasks (typically administrative in nature) which are draining rather than energizing. When this situation develops it is crucial to recall the original source of inspiration for taking charge of the math circle and find ways to regain this role.

*Math circles endure
when directors con-
tinue to find the chal-
lenge of organizing
them year after year
to be stimulating and
rewarding.*

First dilemma

For example, a person who enjoys presenting sophisticated topics to a tight knit, internally motivated group of students will face a challenge when the circle becomes more popular and attracts the attention of parents who bring kids for the sake of improving their math skills. Rather than becoming resigned to watering down material for this new, wider audience, the coordinator must dream up creative means for pursuing their original vision while simultaneously being diplomatic. One possible solution would be to arrange for a “fun group” and a “serious group,” although not in so many words. (They could be dubbed the “Pythagoras” and “Archimedes” groups, for example.) The former group could make soap bubbles with Zome tools, learn basic counting techniques, and meet for a shorter length of time. The latter group, on the other hand, would be expected to complete a challenging weekly problem set, actively take part in a discussion of these problems, and tackle topics such as inversion or generating functions. To avoid doubling administrative duties, oversight of the fun group could be completely delegated to the parents.

Second dilemma

Another coordinator might take pride in the fifty minutes worth of math games that are run each meeting for the kids. Although this sort of activity requires some advance preparation in terms of assembling and photocopying questions, it is easily justified by the level of enthusiasm and active participation it produces in the students. This person would be dismayed, therefore, if the size of the group doubled or tripled so that preparing for the math games became an unwelcome weekly burden. Rather than discontinuing the games or despairing at what has now become a chore, this coordinator might look for solutions such

as raising funds via the parents to pay a team of graduate students to run the games. Another alternative might be to arrange for the more advanced students to host the games every other week for the rest of the group.

As a final example, consider a coordinator who initiates a circle at his university to serve kids who are interested in math but who have not had much exposure to extra-curricular topics. This person would be understandably distressed if the math circle caught on among nearby private schools so that kids with a much broader mathematical background began to dominate the discussions and scare off the target audience. One practical solution would be to change location in such a way that the event simply becomes much more accessible to the target audience. If feasible, the original circle could continue for all interested students under the leadership of another person, while the founding coordinator could establish a second circle at the new site.

All of these illustrations are meant to suggest that a little healthy self-interest on the part of the coordinator is good for the life of the circle and will help to ensure that it can weather the inevitable changes in circumstances that will arise. There are certainly other factors which also contribute to the longevity of a math circle, such as a tight-knit leadership team, supportive parents, or stable funding. But a math circle is often initiated through the vision of one or two people, and to persist (at least in its original incarnation) it is important that these individuals are able to keep sight of their original motivation as the math circle develops over the years. In other words, math circles endure when directors continue to find the challenge of organizing them year after year to be stimulating and rewarding.

Third dilemma

Enduring math circles

Chapter 4

Leading a Math Circle

This chapter, addressing the fine art of preparing and presenting a topic at a math circle meeting, is the final one within this part of the handbook because the author has reasonably chosen to present issues more or less in the order in which a coordinator must handle them. Planning how to use the time during which the students are actually assembled is one of the last considerations on an organizer's checklist, coming to the fore only a day or two before the math circle convenes. But if the chapters were arranged in order of importance, this one would be the first. One instinctively recognizes this as fact: a math circle exists to captivate and inspire the kids, and from their point of view it is defined by what transpires while they are present. So the time has come, belatedly, to discuss the finer points of leading a math circle.

The most important chapter

4.1 Premium presentations

As the person at the front of the room, it is important to keep in mind the fundamental goal of a math circle session: to engage the kids. Capture their interest with an unexpected result. Challenge them with a series of interesting problems. Brainstorm as a group, or have students work individually while circulating to monitor progress. List good ideas, engender debate, and allow participants to present and defend their solutions. All of this activity should take place within a structured environment, of course; providing a framework for exploration is one of the responsibilities of the leader. Ideally a math circle gathering will resemble a kindergarten classroom more than a college lecture.

Keeping things lively

Speaking of lectures, the astute reader will have observed by now that the words 'talk' and 'lecture' have been avoided entirely in this handbook when referring to a leader's activities at a math circle meeting. This is not to say

*A moratorium
on lecturing*

that lectures have no purpose; their format provides one means of efficiently transmitting information from an expert to an assembled group of recipients of the knowledge. However, as Mark Saul (a long-time advocate and promoter of math circles) put it, “The objective of a math circle should never be to cover material. One develops material because it is interesting. It’s an adventure without a definitive goal, necessarily.” Tom Davis, one of the coordinators of the San Jose Math Circle, has this advice to give at his web site¹, “Try not to lecture. Even though introducing new theory and techniques is an integral part of math circles, your sessions should be as interactive as possible. Score yourself: 1 point per minute you talk; 5 points per minute a student talks; 10 points per minute you argue with a student; 50 points per minute the students argue among themselves.”

*Participants must
become intellectually
invested in the pro-
ceedings before they
are able to realize a
return on the math.*

*Getting off on
the right foot*

Effective math circles can follow many different formats, and speakers can employ a wide variety of presentation styles. Here are a few common elements of the most successful sessions. To begin, participants must become intellectually invested in the proceedings before they are able to realize a return on the mathematics being presented. So rather than beginning with definitions or background material, a presenter would be better served by finding a creative means of leading students into the topic for the day. This might take the form of a game or similar activity, such as the one used in the upcoming chapter on binary. Or it might be an elementary yet enticing problem which quickly leads to deeper waters. A group exploration can do the trick: which numbers from 1 to 100 can be written as the sum of two squares, and what can be said about the resulting list? Regardless, an engaging pathway into the material makes for a great way to begin the day.

*Let students do
the work*

Once the pump has been primed a presenter has a good deal of latitude in unpacking the topic for the day. There may be definitions to develop, examples to explain, or notation to agree upon. There is a simple maxim to adhere to that will keep this process lively: make students do as much of the work as possible. This is such sound advice that it is worth emphasizing: *make students do as much of the work as possible*. Rather than define a graph (from graph theory), have a volunteer offer a definition; then play devil’s advocate by drawing in multiple edges, loops, graphs with no edges, and so on to force the group to think carefully about the object they are describing. Similarly, instead of simply presenting summation notation, ask if anyone knows of a compact way to write the sum $4+9+16+\cdots+121$. Find out what they already know! Then challenge

¹Taken from <http://www.geometer.org/mathcircles/>

them to come up with alternate answers, so that they appreciate that

$$\sum_{n=2}^{11} n^2, \quad \sum_{t=0}^9 (t+2)^2, \quad \text{and} \quad \sum_{a=1}^{10} a^2 + 2a + 1$$

are all legitimate answers. Getting students involved in this manner has several benefits: the participants continue to be invested in the unfolding mathematics, while the presenter can assess the level of background of the audience at the same time. Perhaps most importantly, this policy helps to prevent a speaker from inadvertently slipping back into a lecture style and thereby reducing any momentum that had built.

At the heart of any great session lies a selection of great math problems. The way in which a leader chooses to organize and present them is dictated by many factors, including the topic and personal taste, among other things. Thus one person holding forth on induction might choose to illustrate the technique with the entire group on a delightful sample question, then distribute a sheet

Cool math is not enough. One needs cool math packaged properly.

of problems covering a range of subjects and difficulty levels with which to occupy the remainder of the time. Students might vote on which problems sound interesting or which ones are likely to be difficult before they dive in. The leader (and assistants!) could then work with students individually to listen to ideas and offer

hints (never solutions) as appropriate. Periodically the group could reconvene to brainstorm, present solutions, or take a break. Another person might opt to lead the group through a series of carefully chosen problems which lead up to a single main result, such as the theorem on which integers can be written as the sum of two squares. This person might decide to hand out the list of problems only at the conclusion of the exploration in order to keep the group together. A third format for occasional use involves incorporating the questions as part of an exciting team game. This option requires a fair amount of organization and planning ahead, but can be great fun and stimulate focused problem-solving.

A survey of students attending the Stanford Math Circle revealed that they were most satisfied with the meetings in which the presenter reached some sort of concrete conclusion by the end of the session. This finding suggests that simply assembling a sheet of nice problems is insufficient. Paul Zeitz of the San Francisco Math Circle asserts that, “Cool math is not enough. One needs cool math packaged properly.” So rather than simply setting kids to work on an assortment of problems, it would be far more effective to select the problems around a common theme, introduce the theme in a creative manner, have the audience identify several problems that especially appeal to them, then work

Why to put students to work

Presenting nice problems in a strategic fashion

Avoid aimless math

towards understanding those problems as a group. This sentiment also informs the choice of topic to some extent: the most suitable topics are those which can lead to some satisfying result within the allotted time.

There is one final aspect to presenting mathematics at a math circle meeting that is hard to over-emphasize. (But I will try.)

Teach students how to ask good mathematical questions.

*The neglected
critical skill*

Students almost never have the opportunity to ask, “What if ...?” types of questions at any point during their secondary school careers. They have no idea that one of the most crucial skills acquired by a professional mathematician is the ability to ask productive questions; the sorts of questions that lead to new areas of research. All of their training suggests that mathematics is synonymous with solving problems; very few of them stop to wonder where the problems come from. Moreover, working on problems can become tiresome or frustrating. But contemplating new directions to explore, free from the burden of needing to answer all the questions that might arise (at least for the time being), is a marvelous, creative endeavor. Every student should be given the chance to practice this process.

*There is some merit
to selecting a topic
which is not within
one’s standard reper-
toire.*

*Helping students
to ask questions*

So take a few minutes after wrapping up a nice problem to point out that the book is not yet closed on this particular idea, and ask students where it might lead. At first students may need a lot of coaxing. What happens if one uses different numbers or shapes? Is there an analogous result in higher dimensions? What if we allowed three people to play this game, how would that look? Encourage any ideas or attempts; quite often once the first question or two is tentatively offered the floodgates are opened. Help students refine vague ideas into well-formulated questions. This activity can be as rewarding for the leader as for the students—it is exciting to see what they come up with, and invariably everyone leaves with new ideas to pursue.

4.2 Transcendent topics

There is really only one hard and fast rule when it comes to choosing suitable material for a math circle. Speakers should select topics which they find to be fascinating. The kids will quickly pick up on the extent to which a speaker is interested in the proceedings and react accordingly. It is not even necessary that the speaker be especially familiar with the topic, at least prior to preparing

◦ ◦ ◦ ◦ ◦ ◦ ◦
◦

Circle Snapshots

◦ ◦ ◦ ◦



Name: The Washington University Math Circle ◦
Location: Washington University, St. Louis, MO ◦
Director: Blake Thornton ◦
Email: blake@math.wustl.edu ◦
Meeting time: Sundays 1:00–2:00pm ◦
Web site: www.dehn.wustl.edu/~blake/circles/ ◦

A little over five years ago Jennifer Jeffrey discovered an uncommon solution to a relatively common problem. Her son and several of his friends felt limited by their math classes at school and asked if Jennifer could find further mathematical enrichment for them. Since her career involved college publishing, she had the audacity to do what few other parents would consider: she called up Steven Krantz, the chair of the math department at Washington University, and asked if he would be interested in helping to organize an event for pre-college students. To her surprise he agreed, and this unexpected team coordinated a math circle quite successfully for five years. Steve has now taken up a new post at the American Institute of Mathematics while Blake Thornton has taken over his role in the circle's sixth year.

The group originally met late in the afternoon on Wednesdays, but as the students entered high school this time slot began to conflict too heavily with extracurricular activities. The participants overwhelmingly voted to continue the circle, settling on an early afternoon meeting time on Sundays. They also voted for pizza to be delivered following each session, a tradition which still continues. (The snack factor is not to be underestimated.)

This math circle has had to overcome other obstacles as well. When it seemed that the talks were being pitched at too high a level, Steven took a more active role in helping presenters find suitable topics that were accessible to middle and high school students. Getting the word out to parents and teachers is also a perennial challenge. Jennifer is positive that there are a couple kids in every grade level who would benefit from their program, but many families and schools are intimidated by or put off by a university-sponsored activity. Jennifer's solution is to talk to individuals in person to describe how much students enjoy the math circle.

The Washington University Math Circle operates on a tuition model; participants contribute \$150 per semester to attend. This money goes towards pizza, honorariums, and other administrative costs. But there is also a strong element of volunteerism that helps make the math circle a success. As one parent put it, "Math circles are such a great program. There's nothing else out there that touches this."

*Choosing a
great topic*

for the big day. In fact, there is some merit to selecting a topic which is not within one's standard repertoire, particular for speakers with a strong general mathematical background. Not only will the person leading the session enjoy learning a topic about which she has always wanted to know more, but she will also be much better able to view the material from the perspective of a student who has not encountered it before. The net result is a more enthusiastic and accessible presentation. To be sure, there are disadvantages to delivering new material, the most obvious being greater prep time, more hunting necessary for assembling a set of good problems, and a lesser degree of expertise when responding to questions. The latter objection is not really an issue (it's very informative for students to observe how mathematicians respond to questions to which they don't already know the answer), and the first two can be overcome in a couple hours time.

*There are beautiful
motivational ideas
behind even advanced
material that can
form the basis of an
accessible math circle
session.*

*Adapting
advanced
material*

There are ways for adapting almost any topic so that it will work well within a math circle setting. One of the more common types of modification necessary is that of bringing an advanced topic within reach of the target audience. This is the dilemma faced by a research mathematician who wishes to promote his specialty to a general audience. However, there are beautiful motivational ideas behind even advanced material that can form the basis of an accessible math circle session. Moreover, it is important for students to begin to gain a sense of the various fields of mathematics even at the middle school level. Thus a topologist could assemble an entertaining collection of concrete ways to tell the difference between a torus and a sphere. (For example, given five points on the surface of a torus, it is possible to connect every pair of them with a path such that none of the paths cross. This cannot be accomplished on a sphere.) An analyst might spark a rousing debate by asking whether it is possible to cover an open segment with closed segments, a discussion which leads naturally to concepts of infinity, measure, and the Cantor set. Group theory can be brought within reach of eighth graders via symmetry groups. And there are many more variations on this theme.

*Focusing a
broad topic*

Another standard challenge facing speakers concerns how to handle a topic which, despite being readily accessible to students, is much too broad to communicate in one sitting. Topics such as counting, probability, graph theory, number theory, and geometry all fall into this category; topics on which entire books have been written. The temptation to be avoided in these cases is to cover too much material in the given time. Students will either become bored or lost; neither state is optimal. A better approach would be to tightly focus on one

or two inviting problems within the subject that will hold students' attention, then fill in necessary background material when they are motivated to absorb it. For example, rather than lay out the foundations of vector geometry, have the group puzzle out the following fact. If one were to begin at any of four desks in the room, walk halfway towards another desk, continue from there a third of the way towards a third desk, and then head a quarter of the way towards

Students will either become bored or lost if too much material is covered; neither state is optimal.

the final desk, one would arrive at the same finish point, regardless of the order in which the desks were chosen. This strategy of focusing on specific, captivating problems is employed in the subsequent chapters on Sneaky Segments and Making Change.

Finally, many math circles devote at least several of their sessions to preparation for various regional and national mathematical events. For example, most cir-

cles near San Francisco take time to go over general problem solving strategies and proof techniques shortly before the Bay Area Math Olympiad. Other math circles might spend a day helping students get ready for the AMC contests, if there is enough interest. Still other math circles choose to motivate their material solely through the intrinsic beauty of the subject without reliance upon an element of competition.

Over the years an informal canon of math circle topics has gradually developed. Each of these topics is accessible to middle and high school students but can be taken well beyond their standard curriculum. They encompass a multitude of fascinating problems and lend themselves well to engaging presentations. While no list could claim to be comprehensive, this one attempts to include all of the more popular math circle topics. A few general topics are shown in boldface, followed by related subtopics. Further information on them is spread across dozens of sources, but a couple of references cover a significant percentage of them; these are mentioned following the list.

Math Circle Topics

parity	the game of Nim	game theory
Pigeonhole Principle	the fourth dimension	invariants
coloring in math	problem solving	proof techniques
graph theory	tournaments	Ramsey numbers
torus/Klein bottle	Möbius strips	surfaces
cardinalities	infinite series	bijections
induction	sequences	recursion
golden mean	Fibonacci numbers	complementary sequences

Using captivating problems

Circle sessions as olympiad prep

Beaucoup de topics

roots and coefficients	interpolation	the “cubic” formula
generating functions	inequalities	optimization problems
Number theory	primes/divisors	square numbers
perfect numbers	Pythagorean triples	Diophantine equations
binary	triangular numbers	continued fractions
congruences	sums of squares	Euler’s phi function
RSA encryption	infinite descent	quadratic reciprocity
partitions	Farey sequences	Euclidean algorithm
Geometry	length/area	ratios/mass points
points in a triangle	circles/angles	the Arbelos
classic theorems	constructions	paper folding
tiling/tessellations	vector geometry	inversion
Feuerbach’s theorem	projective geometry	homogenous coordinates
similarity	complex geometry	hyperbolic geometry
solid geometry	Euler characteristic	Zome tools
scissors congruence	geometric combinatorics	finite geometries
Counting/Probability	binomial coefficients	Pascal’s triangle
expected value	gambling math	random walks
inclusion-exclusion	Burnside’s lemma	Polya counting
Catalan numbers	combinatorial identities	combinatorial proof
Algebra	symmetry groups	permutation groups
Rubik’s cube/puzzles	Gray code	elliptic curves
matrices	linear programming	finite fields

Resources

The book *Mathematical Circles (Russian Experience)* by Fomin, Genkin, and Itenberg, published by the American Mathematical Society, walks through many of the topics listed above and contains an extensive sampling of problems. Some math circles post handouts at their web sites; the Berkeley Math Circle has a particularly large collection available at <http://mathcircle.berkeley.edu/>. Tom Davis, who has been involved with math circles in the Bay Area for years, has posted notes on many topics at <http://www.geometer.org/mathcircles/>. There are many other useful books and web sites; for a more complete list of resources visit the new web site (still under development as of this writing) at <http://www.mathcircles.org/>.

Purpose of the modules

Each chapter in the second part of this handbook expands one of the above topics into a full-fledged session. There are detailed presentation notes, two sets of problems to accompany each topic, background, and hints/answers. The purpose of this compilation is two-fold. Most importantly, these chapters should give speakers a sense of how to transform a promising topic into a lively exploration for a group of motivated kids. The modules span a range of difficulty

levels and subject material in order to be as helpful as possible in this regard. On the other hand, they are not meant to serve as a sample syllabus for a semester's worth of meetings.

Of course, in a pinch they can also serve as ready-to-go presentations; do not hesitate to mimic the suggested dialogue or use the problems as they stand—they were assembled partly with this purpose in mind. (Just please mention the source!) Most of the sessions described will need editing before use, depending

The upcoming chapters give a sense of how to transform a promising topic into a lively exploration.

on the audience, time available, and leader's mathematical expertise. Select the most appropriate problems and pick and choose from the presentation notes; there is usually more material than will fit into a single meeting. The stars underneath the titles gauge the approximate level of mathematical maturity necessary on the part of the participants. One star topics are suitable for most middle school groups, two star topics

Using the sample presentations

could work for an advanced middle school or normal high school crowd, and three star topics should be reserved for a strong high school audience.

All these sample presentations and more should soon be available at the web site <http://www.mathcircles.org/> maintained by the Mathematical Sciences Research Institute. In particular, the L^AT_EX code for the problem sets appearing in the upcoming chapters may be downloaded, which should save typing time. The goal of that web site, similar to that of this handbook, is to serve as a resource for coordinators and leaders of math circles. The specific mission and function of the site will undoubtedly evolve over time, so please visit the web site for more information. And now, on to the mathematics!

The MSRI web site

Part II

Presentations

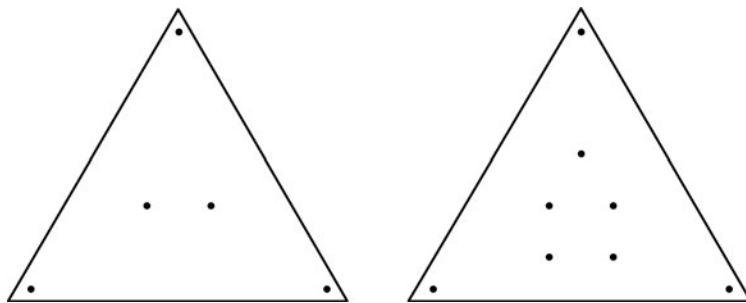
Chapter 5

The Game of Criss-Cross

Euler Characteristic (★★)

Overview. The regions on a map and the faces of a cube both illustrate a very natural sort of situation: they are each examples of regions that are joined together by common boundaries. The Euler characteristic is a quantity that describes how these regions fit together. (More precisely, it describes the sort of surface on which these regions reside.) Because the Euler characteristic depends only on the manner in which the regions meet along their boundaries, rather than on their precise shape, it is a topological concept. It is an elegant idea and powerful tool that middle and high school students can readily discover on their own. As such, it makes an excellent math circle topic for any level. One of my favorite ways to develop these ideas is through a simple game.

Activity. The game of Criss-Cross is played on a blank sheet of paper by two players. The game board is created by drawing three points at the vertices of a large equilateral triangle, along with two to seven additional points anywhere in its interior. Two sample boards are shown below. Players alternate turns drawing a single straight line segment joining any two points, as long as the segment does not pass through any other points or segments already appearing on the game board. The winner is the last player able to make a legal move.



Two sample Criss-Cross boards

Problems.

1. How many different moves can the first player make on the game board pictured at left above?
2. Will the first or second player win on the left-hand game board? Explain why any game on this board always lasts for the same number of moves.
3. Play three games of Criss-Cross using the right-hand game board. Make a conjecture regarding the outcome of any game played on this board.
4. By trying games on several different boards, come up with a method of predicting the winner of any game of Criss-Cross based on the board configuration.
5. For each of the games played in the previous problems, count the number of vertices (points), edges (segments), and faces (regions) appearing in the completed game board. Be sure to include the area surrounding the game as one of your regions. For example, you should find that the left-hand game board above has five vertices, nine edges, and six regions.
6. Compare the number of edges and faces on each completed game board, then make a conjecture about these two quantities. Finally, prove your conjecture. (HINT: first explain why every region on a completed game board is triangular. Then imagine cutting out all the regions with a pair of scissors. How many edges must have appeared on the completed game board, in terms of F ?)
7. The expression $V - E + F$ is known as the Euler characteristic. Prove that the Euler characteristic of any completed game board is equal to 2.
8. Use the relationships between V , E , and F developed above to predict the number of edges and faces that will appear on a completed game board which starts with a total of 99 points. Will the first or second player win this game?
9. Use reasoning similar to the previous problem to prove that your method of predicting the winner on any game board is valid.

Presentation Notes. Begin by introducing the game of Criss-Cross to the entire group. Describe how the game board is created and draw a board with only two interior points to keep the demo game short. Invite two students to play the game, explaining the rules in the process. Note that the first player won the game, and invite the students to explain why this always occurs on this game board. They should point out that there are ten possible segments that can be drawn, but two of them intersect. Hence there will always be nine segments on a finished game board, meaning that the first player wins. Don't erase the game yet since you will return to it later in the circle.

Next have students play a game or two with their neighbors. Mention that students should not draw in the edges of the outer triangle as part of the original game board; these will be filled in later during play. It is advisable to avoid placing three points in a line as much as possible. It is also helpful to draw solid, visible dots for points. It is permissible for players to draw slightly curved segments for the sake of clarity, since this will not affect the outcome of the game. Direct them to record whether the first or second player won in each case and to save their games for later analysis.

When enough games have been played (a total of twenty is more than sufficient), restore order and ask for their ideas regarding how to win at Criss-Cross. If necessary, guide them by creating a chart which tabulates the total number of points on the game

board in one column and the player who won that particular game in an adjacent column. Ultimately students should make two conjectures: that the winner of a game depends only on the number of points on the board rather than on the precise position of the points or segments drawn, and that the first player will win if there are an odd number of total points, while the second player will win in the even case.

One approach to understanding the conjectures that were just made lies in counting the number of faces (regions), edges (segments), and vertices (points) on the game boards. Illustrate these counts using the demo game from earlier, in which there were five vertices, nine edges, and six regions. (The entire area surrounding the triangular game board should be counted as a region. It will soon be apparent why it is important to include this region.) Summarize this data succinctly underneath the game board by writing $V = 5$, $E = 9$, and $F = 6$. Have students make similar counts for their game boards, then display all the results in an accessible location, perhaps on a large table or taped to the wall. Allow students to survey the data and make observations.

When students have had enough time to make and test conjectures, reconvene and write down any relationships the students discovered among the numbers associated with each game board. Hopefully they will have noticed that the ratio of edges to faces is constant; more precisely, that $\frac{E}{F} = \frac{3}{2}$, which can be written more neatly as $2E = 3F$. If not, suggest that they look more closely at how the number of edges compares to the number of faces. Ideally they will have also observed that $V - E + F = 2$ in every case. If needed, help students discover this relationship by inquiring whether there are more vertices, edges, or faces in general. Then ask if the edges outnumber the faces and vertices combined, and if not, by how much. These two observations are the keys to explaining the game of Criss-Cross.

Depending on the available time, interest, and problem-solving level of the group, you could now give students the opportunity to demonstrate why the two relationships just found must always occur. One could also simply outline a proof of these facts, or relegate them to a problem set, or employ any combination of these tactics. The fact that $2E = 3F$ is probably more accessible, and hinges on the fact that on a completed game board every face (including the outer region!) has a triangular boundary, because otherwise another segment could have been drawn. Imagine cutting out all the faces, and counting the edges among the resulting pieces; clearly the total will be $3F$. On the other hand, every edge is counted twice in this process, since every edge borders exactly two faces. (Now it becomes apparent why it is important to include the outer region as one of the faces.) It follows that $3F = 2E$.

The expression $V - E + F$ is known as the Euler characteristic, and its value will always equal 2 for any connected, planar graph. (I.e. a network of vertices joined in pairs by non-intersecting edges in which any two vertices are connected by a sequence of edges.) One way to establish this result is to begin with a single vertex, for which $V = 1$, $E = 0$, and $F = 1$, so $V - E + F = 2$ holds. Then consider the various ways of including more edges. One could branch out by adding one new vertex, then joining it to the current configuration by an edge. Alternatively, one could join two existing vertices by an edge, which increases the number of faces by one. Argue that in either case the quantity $V - E + F$ remains unchanged, and that any connected planar graph may be obtained in this manner.

We are now in a position to prove that the conjecture students made earlier regarding how to predict whether the first or second player will win at Criss-Cross based on the parity of the number of points on the game board. Have them determine E and F if $V = 8$, for example. Presumably students will have the algebraic background to solve the system of equations $2E = 3F$ and $8 - E + F = 2$, or at least substitute

$E = \frac{3F}{2}$ and $V = 8$ into the equation $V - E + F = 2$, then solve for E . Young students could even be asked to find solutions by trial and error, rather than via algebra. They should compare their results to an actual game involving eight points to check that their answers are reliable. Then have them compute the number of faces and edges on a completed board with 99 initial points, and thus predict the winner of such a game. Ultimately, students should deduce algebraically that $E = 3(V - 2)$. Hence if V is even, then so is E , implying that the second player will win. Conversely, if V is odd then E will be also, leading to a win for the first player.

There are many possible avenues of exploration from here, as suggested by the further problems below. One could consider the effect of using a square or other polygon as the outline of the game board, rather than an equilateral triangle. Or the discussion could move on to the Euler characteristic of polyhedra, where the terms ‘faces,’ ‘edges,’ and ‘vertices’ are more natural. There is also a nice application of the Euler characteristic to prove that the complete graph on five vertices is not planar. For younger groups, the above material is probably more than sufficient by itself.

Further Problems.

1. In a certain small country there are villages, expressways, and fields. Expressways only lead from one village to another and do not cross one another, and it is possible to travel from any village to any other village along the expressways. Each field is completely enclosed by expressways and villages. If there are 100 villages and 141 expressways, then how many fields are there?
2. Suppose that we change the outer boundary of a Criss-Cross board so that it consists of four points at the corners of a square. Play games in which there are either one, two, or three additional points in the interior of this square. Based on the outcomes, make a conjecture regarding how to predict whether the first or second player will win on a square board.
3. Explain why $3F + 1 = 2E$ on a completed game board with a square boundary.
4. Suppose there are a total of 99 points on a Criss-Cross board having a square boundary. How many edges will appear in the completed game? Will the first or second player win?
5. Prove your conjecture from above as to who will win on a square game board. Then extend your result to game boards with pentagonal boundaries and beyond.
6. A polyhedron is the three-dimensional analogue of a polygon. It is a solid all of whose faces are polygons, such as a cube, tetrahedron, or triangular prism. For each of these three examples, compute the number of vertices, edges, and faces in the polyhedron. Confirm that the Euler characteristic equals 2 in each case; i.e. that $V - E + F = 2$ holds for all three polyhedra.
7. Assume for now that $V - E + F = 2$ for any connected planar graph. (This is a network of vertices joined in pairs by non-intersecting edges in which any two vertices are connected by a sequence of edges. A completed Criss-Cross game board is an example of a connected planar graph.) Use this fact to demonstrate that the Euler characteristic of any polyhedron must also equal 2.
8. A certain polyhedron is built entirely from triangular faces in such a way that five faces meet at each vertex. How many faces will such a polyhedron possess? (HINT: first deduce that $3F = 2E$ and $3F = 5V$.)

9. In another polyhedron the same three types of faces meet at every single vertex; namely, a square, a hexagon, and a decagon (10-sided polygon). How many vertices must appear in this polyhedron?
10. Find a way to position four points on a sheet of paper so that when every pair of points is joined by a straight line segment, none of the segments intersect. (There will be six segments in all.)
11. Plot five points on a sheet of paper and draw a segment connecting every pair of points. How many edges are needed? Experiment a bit to decide whether it is possible to position the points so that none of these edges cross.
12. Suppose that someone claims to have a diagram which solves the previous problem. Even though you can't see their picture, what must be the values for V , E , and $V - E + F$? From here deduce the number of faces F in their diagram. (Recall that the region surrounding the diagram counts as a face.)
13. Prove that $3F \leq 2E$ for any connected planar graph. Use this relationship to prove that the numbers in the previous problem are contradictory, meaning that the hypothetical diagram cannot really exist.

Hints and Answers.

1. Since the expressways don't cross the map of the country will form a planar graph with $V = 100$ and $E = 141$. The Euler characteristic is 2, hence $F = 43$. But one of these regions surrounds the entire country, so there are just 42 fields.
2. You should find that now the first player wins if there are an even number of total points, while the second player wins when there are an odd number of points.
3. All of the regions on a completed game board will still be triangles, except for one: the outer region now has a square boundary. Upon cutting out all the pieces there will be one more edge than if all the regions had been triangles, hence the $3F + 1$.
4. Solving $99 - E + F = 2$ and $3F + 1 = 2E$ we find that there will be 290 edges, hence the second person will be the last one to be able to make a valid move.
5. In general, eliminating F from the equations $V - E + F = 2$ and $3F + 1 = 2E$ yields $E = 3V - 7$. Thus E will be even when V is odd and vice-versa, as desired. In general, one should find that a game board has V points, B of which are boundary points, then the first player wins when $V + B$ is even, while the second player wins when $V + B$ is odd. More succinctly, the outcome of the game depends only on the number of interior points!
6. The expression $V - E + F$ becomes $8 - 12 + 6$, $4 - 6 + 4$, and $6 - 9 + 5$ for a cube, tetrahedron, and triangular prism, respectively.
7. Imagine turning the surface of a polyhedron into a planar graph by poking a hole in one of the faces, then "flattening" the rest of the surface by widening the hole and stretching the surface flat. The number of vertices, edges, and faces remains unchanged during this process.
8. We establish $3F = 2E$ just as before. Now cut out each of the faces and count the total number of vertices among the resulting pieces, giving $3F$. But each vertex is counted five times, hence $3F = 5V$. Now combine these equations with $V - E + F = 2$ to deduce that $F = 20$. (So the polyhedron must be an icosahedron.)

9. Suppose there are F_1 squares, F_2 hexagons, and F_3 decagons among the faces. First argue that $4F_1 = 6F_2 = 10F_3 = V$ by counting the total number of vertices in three different ways. Cutting out faces and counting edges leads to $4F_1 + 6F_2 + 10F_3 = 2E$. Finally, we also have $F = F_1 + F_2 + F_3$ and $V - E + F = 2$. This is enough information to solve for the unknowns, yielding $V = 120$. (The polyhedron is a beautiful Archimedean solid known as a truncated icosidodecahedron.)

10. The vertices of a square won't work, but a point inside a triangle does the trick.
11. There will be ten edges, but some pair of them always crosses.
12. We know that $V = 5$, $E = 10$, and $V - E + F = 2$, hence $F = 7$.
13. If all faces are triangles, then we already know that $3F = 2E$. In general, if the faces have $F_1, F_2, F_3, \dots$ edges then we have

$$2E = F_1 + F_2 + F_3 + \dots \geq 3 + 3 + 3 + \dots = 3F.$$

But then we would have $2(10) \geq 3(7)$ for our hypothetical graph, which is a problem.

Chapter 6

Double Time

Binary representation (★)

Overview. At one level, binary is just arithmetic performed with only the two digits 0 and 1, as opposed to the customary ten digits 0, 1, 2, ..., 9. Since students are already familiar with performing calculations in our standard base ten notation, they can enjoy discovering the process for adding and multiplying binary numbers without requiring much direct instruction. As students will find, binary arithmetic is based on powers of two, so binary is directly or indirectly related to almost any process these powers. Examples abound and include efficient systems of weights, programming algorithms, and many popular mathematical puzzles, such as the Towers of Hanoi. I like to introduce this concept to math circle students through an exercise which helps them to really get the feel of binary.

Activity. Any group of four people can take part in this activity, called “Last One Standing.” The four individuals should sit in a row of four adjacent seats, all facing the same direction. If possible, they should orient themselves as if they were standing in a line, so that the person in the rear can see the other three, while the person in front can’t see anyone else. It is also fine to sit shoulder to shoulder, all facing outward, as if seated on a couch. Just replace ‘in front’ and ‘behind’ with ‘to the left’ and ‘to the right’ in these directions.

To begin, all individuals should be seated. The object is to arrange for the last person in line to be standing, while all others are seated. The participants can move according to a single rule: a person can change state (by standing up or sitting down) if the person immediately in front of them is standing, while all others in front of them are seated. Otherwise they are locked in position and may not move. The status of people behind them does not matter. The person in front, to whom this rule does not apply, can stand or sit at will.

Problems. During the math circle students will consider some of the following questions pertaining to the “Last One Standing” activity. They can then move on to other questions related to binary numbers and arithmetic.

1. What is the least number of steps necessary to accomplish the task?

2. If we allow more than four people to sit in a row, then what is the fewest number of moves needed to have the last person standing and all others seated?
3. How many times does a particular person in the row have to move in the course of an optimal solution?
4. How long would it take a group of twenty people to complete the activity?
5. Based on the pattern, determine the next ten “numbers” in the sequence

1, 10, 11, 100, 101, 110, ...

6. What is the 75th number in this sequence? What about the 99th number?
7. Where in the sequence does the number 101010 appear? How about 1010001?
8. Find the 75th number in this sequence by adding 11001 (the 25th number) and 110010 (the 50th number).
9. Find the 99th number in this sequence by multiplying 1001 (the 9th number) and 1011 (the 11th number).

Presentation Notes. Introduce the “Last One Standing” activity by having four volunteers sit in a row at the front of the room or some other conspicuous location. Explain the rules of the activity, then allow the group to attempt the task. Encourage the rest of the students to maintain a respectful silence as long as the volunteers are making progress. However, be sure to immediately interrupt their efforts if an illegal move is made, then ask the audience where they went astray and what move should have been made instead. Applaud your brave volunteers when they finally successfully accomplish the activity.

It is rather important to rehearse this activity yourself ahead of time. For example, you can use the index and middle fingers of each hand to simulate the four individuals. The first person will stand, which allows the second person to stand. Next the first person sits back down, so that the third person can stand. Then the first person stands again, permitting the second person to sit, and so on. The entire activity will require fifteen steps and have a distinctly binary feel to it.

Then divide all the students into groups of four and have them practice on their own until they are able to finish the task smoothly. (Leftover students can be assigned to existing groups to act as coaches or periodically rotate into the lineup.) When a group is ready to demonstrate the steps, surprise them by pulling out a stopwatch and timing their solution. I would recommend giving each group at most two attempts to get the last person standing; if they make an invalid move they have to start over from the beginning. Keep track of the best times at the front of the room until just before students begin to tire of the activity. Then award a prize to the fastest group and return the room to its former state.

Next instruct the participants to ask mathematical questions about the activity that just took place. This is one of the most important parts of the math circle, since it gives students the opportunity to engage in a fundamental aspect of any mathematical investigation; namely, asking fruitful questions. You will probably need to be patient, since students don’t have much experience with this process. If needed, direct them to think about what quantities related to the activity could be counted. Ask them if they can generalize their questions. Given time, they should be able to pose questions

similar to the first four stated above. I typically include “How long would it take a group of twenty people to complete the activity?” in their list because it gives a good focus to the discussion and has an unexpected answer.

Students can proceed to find answers to the questions they just posed. Emphasize that one of the best ways to solve hard problems is to formulate and answer easier versions of the question or examine simple special cases. For example, this strategy could be employed for finding the number of steps required for a group of twenty to complete the activity. A table which summarizes this data might look like the one below.

group size	1	2	3	4	5	6
steps needed	1	3	7	15	?	?

Students can then find patterns to predict the next couple of entries in the table. Responses usually include “Double each number and add 1 to obtain the next number,” or “The difference between successive entries is always a power of 2,” along with less helpful ideas. Someone might also notice that the entries in the bottom row are all 1 less than a power of 2. (If not, steer them to this observation or make it yourself.) Compare and contrast these ways of looking at the data, and discuss the advantages and disadvantages of each. Thus the first idea provides a recursive way of generating entries and provides a natural way of understanding the pattern. On the other hand, the last idea gives an explicit formula and is more useful for computing the twentieth entry. Stress that finding patterns is only the beginning of the story; the real mathematics lies in proving that our patterns persist. While noticing the patterns will allow us to compute answers, the underlying mathematics contains the beautiful ideas that make the whole enterprise worthwhile.

By this time students will have conjectured that it will take five people 31 moves to get the last person standing alone. Have five volunteers give this a try and count the steps to confirm this prediction. Then try again, this time allowing them to skip straight to the step in which the fourth person is standing, since they are well-acquainted with how to accomplish this in 15 steps. The fifth person can then stand on the sixteenth move. Ask how many more steps it will take to result in the fourth person sitting back down. Once they realize that it will take 15 more steps (same as the initial 15 steps, except the fourth person sits instead of stands at the appropriate moment) then they have essentially grasped the recursive nature of the process. To cement the idea, add a sixth person at the end of the line and ask how many steps are required in this scenario. The answer is 63: 31 steps to have the fifth person standing alone, 1 step for the sixth person to stand, then 31 more steps to seat the fifth person again. This explains the “double and add 1” observation.

At your discretion, use the recursive formula to prove the explicit formula. In other words, suppose we have a number that is 1 less than a power of 2. We wish to explain why doubling this number and adding 1 results in another number that is also 1 less than a power of 2. I usually begin with

$$15 = 16 - 1 \quad \text{so} \quad 2(15) + 1 = 2(16 - 1) + 1 = 32 - 2 + 1 = 32 - 1 = 31,$$

then introduce variables to polish off the general case:

$$2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1.$$

Regardless, at least mention that the recursive formula can be used to derive the explicit formula. At this point the group will be able to calculate how long it will take

a group of twenty to finish the task. (It is entertaining to have them guess first. Most estimates are less than a couple of hours, which is well below the actual duration.) Use the winning time from above to estimate the time needed per move; a half second is a reasonable amount. Using this figure, it would take a little over six solid days to finish, with no breaks for eating or sleeping, at least for the first person in line!

During the remainder of the math circle let the students develop as much binary arithmetic as they are able. The hardest part about leading this activity is to resist the temptation to present it all yourself. One possible sequence of questions is outlined in the problem set above; this has proven to be an effective approach in other math circles. Here are a few ideas to help guide the discussion.

- Discourage students from referring to the binary numbers 10, 11, and 100 as ‘ten,’ ‘eleven,’ and ‘one hundred,’ but call them ‘one-zero,’ ‘one-one,’ and ‘one-zero-zero’ instead.
- Make sure they notice and appreciate the fact that the “special” numbers 10, 100, 1000, and 10000 occur in the second, fourth, eighth, and sixteenth positions in the sequence.
- Based on the previous item, ask for a term of the sequence near the 75th term that they know for sure, then work from there. Help the group appreciate that $75 = 64 + 11$ translates into $1001011 = 1000000 + 1011$.
- In general, finding the powers of 2 used to “build” a number gives a shortcut to finding a term in the sequence. For instance, since $99 = 64 + 32 + 2 + 1$ we deduce that the 99th term will be 1100011.
- Try this algorithm in reverse in the next question to find that 101010 is the 42nd term and 1010001 is the 81st term.
- Gradually transition from treating the strings of 0’s and 1’s as terms in a sequence to thinking of them as legitimate numbers. Ask what sorts of operations one can perform with numbers (addition, multiplication, etc.) and suggest that we try these operations on binary numbers as well.
- Based on their knowledge of arithmetic, see if the students can figure out a few simple examples such as $101 + 1 = 110$ and $1011 + 111 = 10010$ on their own. Review the process of carrying together, then let them try the next question in the problem set above.
- Similarly, see if they can compute $1011 \cdot 101 = 110111$ before illustrating long-hand multiplication for everyone. They can then check their earlier prediction for the 99th term by answering the final question.

Other avenues of exploration could be based on the problems below, or you could insert your favorite binary activities at this point. For instance, one could discuss how and why computers utilize binary, continue with the Towers of Hanoi puzzle, investigate Gray code, or just wrap up. I find that students are consistently more fascinated by binary than I expected. So have a great time with this topic in your math circle.

Further Problems.

1. List the next five binary numbers after 1001011.
2. Convert the unlucky numbers 13 and 666 into binary.
3. Compute the sum $1110010100101 + 1011011011101$ in binary, writing your answer in binary. (Don't even think about converting them to decimal numbers.)
4. Find a pair of binary numbers, each with exactly three 1's in their binary representations, such that their sum has only a single 1 among its digits.
5. Perform the multiplication $10001001 \cdot 1101001$ completely in binary.
6. Find a four-digit binary number such that placing an extra 1 at the beginning and end results in a new six-digit number which is five times as large.
7. Merchants used to weigh items using a set of standard weights and a balance scale. For example, with nine 1-oz weights and nine 10-oz weights a vendor could measure any amount from 1 to 99 ounces by placing the item on one side of the scale and certain weights on the other side. However, this requires hauling around eighteen weights. Can you figure out how to accomplish the same thing with only seven weights? (In this problem assume that the weights may only be placed on one side of the scale, while the item goes on the other side.)
8. If you found the solution that I'm thinking you did to the previous problem, then your weights total 127 ounces—significantly more than the 99 ounces that our merchant had to carry before. Now figure out how to weigh any amount from 1 to 99 ounces with seven weights having the smallest possible total weight.
9. If we include leading zeroes, then the binary numbers from 0 to 7 look like

000, 001, 010, 011, 100, 101, 110, and 111.

Arrange these eight numbers in a circle so that every pair of adjacent numbers has identical digits except in one spot. (For example, 110 could be next to 010 but not 011.)

10. Once you have solved the above problem, can you think of a shortcut for achieving the same thing with the sixteen binary numbers from 0000 to 1111? (Try to build on the previous solution rather than starting from scratch.)
11. How many ways are there to arrange the binary numbers from 0 to 7 in a circle so that adjacent numbers differ in only one digit? **HINT:** there is a beautiful way to picture this problem using a cube whose vertices have coordinates $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$, $\dots$, and $(1, 1, 1)$.

Hints and Answers.

1. The numbers are 1001100, 1001101, 1001110, 1001111, and 1010000.
2. Written in binary we have 1101 and 1010011010. Not nearly as unlucky-looking as before!
3. The sum comes to 11001110000010.
4. The numbers 10101 and 1011 sum to 100000, for example.

5. The product is 11100000110001.
6. Let N be the sought after binary number. Appending 1's to the front and end of N yields the number $32 + 2N + 1$, since N has four digits in binary. We wish for this quantity to be five times as large as N .
7. The solution that naturally presents itself is to use weights of 1, 2, 4, 8, 16, 32, and 64 ounces.
8. The total weight has to be at least 99 ounces in order to weight an item which is that heavy. There are many ways to find seven weights which total exactly 99 ounces. The most obvious, perhaps, is to simply to replace the 64-oz weight in the previous solution with a 36-oz weight.
9. There are many possible answers. One is

000,001,011,010,110,111,101,100.

10. Place a 0 in front of all the above numbers for the first eight entries around the circle. Then place a 1 in front of the same numbers and list them next, in reverse order. In other words,

0000,0001,...0100,1100,1101,...,1000.

11. The vertices of the cube represent the binary numbers from 0 to 7 via their coordinates. (For example, the vertex at $(1,0,1)$ stands for 5.) The key is to realize that the edges of the cube connect exactly those pairs of vertices which differ in one coordinate/digit. (Draw a picture to see this.) So the question boils down to finding the number of ways to traverse the edges of a cube so that you visit each vertex exactly once and return to your starting point. There are essentially six different ways this can be done; double this if you distinguish the direction in which the path is traced out, and eight times this if you keep track of where the starting point is located.

Chapter 7

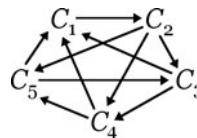
King Chickens

Graph theory (★★)

Overview. As anybody who works with farm animals knows, bringing together a flock of chickens that are not already familiar with one another is a recipe for trouble. A barnyard melee is likely to ensue, in which every pair of chickens will determine which of the two of them is dominant over the other, primarily by pecking. (Hence the origin of the phrase ‘pecking order.’) It is natural to describe a pecking order with a complete directed graph, in which the chickens are represented by points and the relationship between any two chickens is represented by an arrow pointing in the same direction as the pecking. The modifier ‘complete’ refers to the fact that there is an arrow between every pair of points. In general, directed graphs provide a useful way to represent and study any round robin tournament.

Although the formal study of graph theory only began in the first half of the twentieth century, mathematicians have developed a formidable body of knowledge related to this fascinating and versatile subject. This particular development of the concept of directed graphs is based on the delightful article “The King Chicken Theorems,” by Stephen Maurer, appearing in *Mathematics Magazine*, volume 53 (1980), pages 67–80. Besides providing an entertaining backdrop to the subject of graph theory, this approach has the added benefit that students can become involved in the process of formulating mathematical definitions and appreciating how their choices influence the resulting theory.

Introduction. Let us say that a flock consists of one or more chickens, which we will generically label as C_1, C_2, C_3 , and so on. Between every pair of chickens there is a pecking order, which we will indicate by an arrow. For example, we write $C_1 \rightarrow C_2$ if C_1 pecks C_2 . In general, pecking need not be transitive, meaning that if C_1 pecks C_2 , who pecks C_3 , then it is not necessarily the case that C_1 pecks C_3 . One possible pecking order among a flock of five chickens is shown at right.



There are many possible ways to define what we mean by a “king chicken.” In what follows we will say that K is a king if, given any other chicken C , either K pecks C directly (so that $K \rightarrow C$), or else there exists a field marshal F such

that K pecks F , who pecks C (written compactly as $K \rightarrow F \rightarrow C$). Note that a king might have several field marshals, who in turn peck all the remaining chickens among them. For example, in the illustration above, chicken C_1 is a king with only one (very powerful) field marshal. Chicken C_5 is also a king, with two field marshals.

Problems.

1. For each chicken in a flock, count the number of other chickens which that particular chicken pecks. Let K be the chicken with the highest peck count. (If there is a tie, let K be any one of the winners.) Prove that K is a king. This shows that every flock has at least one king.
2. How can we arrange for a flock (of any size) to have exactly one king?
3. If a chicken has the barnyard to itself, of course it is king. How many kings will there be in a flock with two chickens?
4. There are essentially two different possible pecking orders for a flock with three chickens. How many of the three chickens are kings in each case?
5. Find a way for a flock of four chickens to have exactly one or three kings. Then show that it is impossible for such a flock to have exactly two or four kings.
6. Show that if we have a flock of n chickens with exactly k kings, then there exists a flock of $n + 1$ chickens which also has exactly k kings.
7. Construct a pecking order for a flock with an odd number of birds in which every chicken is a king.
8. Suppose we have a flock of n chickens in which every chicken is a king. Explain how to construct a flock of $n + 2$ chickens with the same property; so that every chicken is again a king.
9. Establish the following lemma: given a particular chicken C , if C is pecked by other chickens, then one of the chickens that pecks C must be a king.
10. Use the lemma to prove that no flock can have exactly two kings.

Presentation Notes. It is hard to resist introducing this topic by saying, “Today we’re going to talk about chickens.” Describe the pecking order phenomenon as outlined in the overview (without drawing any examples on the board), then ask how one might represent a pecking order within a flock of three chickens. Develop the concept of a complete directed graph via their responses, and also find both types of pecking orders for flocks of three chickens. (Either each chicken pecks one other in a circular fashion, or one dominant chicken pecks both of the others.) Inquire as to what it means for a pecking order to be transitive. (It is likely that students will not know, but it never hurts to ask.) Explain the concept of transitivity using the pecking orders just illustrated. (The symmetric pecking order is not transitive, while the one involving a dominant chicken is.)

At your discretion, it might be worthwhile to investigate the results of requiring a pecking order to be transitive. Students should discover that there is essentially only one type of pecking order possible with this restriction: that there is a chief chicken who

pecks all the others, then a second in command who pecks everyone but the chief, and so on down to a lowly serf chicken who doesn't peck anyone. In other words, a strict hierarchy results. Because the choice of king chicken is clear, it is more interesting to discuss the idea of king chickens in the non-transitive setting. Incidentally, actual chicken flocks usually do not exhibit completely transitive behavior.

One of the central components of this math circle is engaging students in the process of creating a definition for "king chicken." Announce that you would like them to come up with at least three criteria for what constitutes a king. The most obvious choice is to count the number of others that each chicken pecks, then crown the chicken (or chickens) with the highest peck count as king. But if pressed, students can usually concoct creative alternative definitions. (One such is given in the further problems.) They will probably not propose the definition given above, but help them formulate it, perhaps by saying, "It would be nice if a king K had the property that given any other chicken C , we had $K \rightarrow C$. Of course, there might not be a chicken that pecks every other. But for those chickens which K does not peck directly, what might be the next best thing?" Hopefully they will eventually suggest the idea of field marshals; if not, present this definition along with their others. Wrap up this portion of the circle by stating that we will be investigating the latter definition more closely, since it leads to some interesting and unexpected results. You might also mention that this is one of the yardsticks by which definitions are measured by mathematicians.

It is natural to wonder about the extent to which our definition is equivalent to the others. Help frame the discussion by asking whether students can find an example of a chicken that is a king, but which does not have the highest peck count. In the extreme case, see if they can construct a pecking order in which the chicken with the lowest peck count is a king! (The example in the introduction shows how this can be done.) Next ask if they can arrange for the chicken(s) with the highest peck count not to be king. They will eventually decide that this cannot be done, so conjecture that any chicken with a maximal peck count is a king. Prove this by contradiction: if this chicken C were not king, then there would exist another chicken C' such that $C' \rightarrow C$, and also C' pecks all the chickens that C does. (So that none of C 's field marshals peck C' .) But then C' has a higher peck count than C , contradicting our assumption.

Announce that the remainder of the circle will be devoted to answering the question, "How many kings can a flock of n chickens have?" Ask for theoretical bounds on the number of kings; presumably this figure could range from 0 to n . Perhaps a student will realize that having no kings is impossible, otherwise ask if a flock might have no kings. If necessary, have students consider why the preceding result implies the presence of at least one king. From here, turn the students loose to work on the problem. Make sure that their goal is clear: they are trying to determine the possible numbers of kings for any size flock. Also be sure that the strategy for answering this question is clear: they should generate lots of examples of pecking orders, starting with small size flocks and working up, to see what numbers are possible. Have them make conjectures, prove them if possible, and also keep track of any useful constructions that they discover for building new pecking orders from existing ones.

At an appropriate moment redirect students' attention to the front and take stock of their findings. Many of the possible results are included in the problems above. At the minimum, it is probably worthwhile to show that any size flock can have exactly one king (have one chicken peck all the others) and that flocks with an odd number of birds can have all kings (arrange the n chickens in a circle and have each chicken peck the next $(n-1)/2$ birds proceeding clockwise around the circle). It would also be good to establish that one can always augment a flock of kings by two chickens

to obtain a larger flock of kings; simply have one of the new chickens peck the entire original flock, have the whole flock peck the other new chicken, who in turn pecks the first new chicken. It is a good exercise to confirm that this process does the job.

Students may also conjecture that when n is even, a flock of n chickens cannot consist entirely of kings. This is an appealing conjecture because it happens to be true for $n = 2$ (clearly) and $n = 4$ (only a few cases to check). So it may come as a surprise that it is possible to create a pecking order for $n = 6$ in which all chickens are kings, although it is slightly tricky to find. Coupled with the previous recursive construction, you have now shown that there exist flocks of n chickens, all of whom are kings, for every positive integer n except for $n = 2$ and $n = 4$. This avenue of exploration would make for a nice conclusion to the circle.

If time permits, an even more satisfactory finish would be to establish one other conjecture that students might make; namely, that it is impossible for a flock to have exactly two kings. After confirming that their hunch is correct, state the following lemma which will be helpful in its proof: “Given a particular chicken C , if C is pecked by other chickens, then one of the chickens that pecks C must be a king.” It is helpful to draw a diagram of the flock highlighting the group of chickens that C pecks and the remaining chickens that all peck C . Explain that we want to prove that there must be a king (of the whole flock) in the latter group, and ask how we might find it. If need be, have students consider separately just the subflock of chickens that peck C , and ask what must be true about that subflock. It is not hard to confirm that a king of the subflock is in fact a king of the whole flock, thus proving the lemma.

The proof of the conjecture is now fairly immediate, although students might overlook the first step. The problem is that if a flock has exactly two kings, then one of them must peck the other. The lesser king is therefore pecked by another king, but the greater king is not! The subtlety lies in noticing the caveat in the lemma, which states that it only applies to chickens that are pecked. Ask students why the greater king must be pecked by some other chicken. (Since the lesser king is a king, after all, one of his field marshals must peck the greater king.) So we have reached a contradiction, and we’re done.

Further Problems.

1. Suppose that pecking were transitive, meaning that given any three chickens, if $C_1 \rightarrow C_2$ and $C_2 \rightarrow C_3$, then $C_1 \rightarrow C_3$. Show that in this case there must be some chicken who pecks all the others.
2. Find a flock of six chickens in which every chicken is a king.
3. If a flock has exactly one king, prove that he must peck all the other chickens.
4. “Let K be a king which is pecked by some other chicken. Then there must be a king among the field marshals of K .” Is this statement valid for every pecking order? Either prove this assertion or find a counterexample.
5. Here is an alternate method for defining king chickens. Let us say that to “cull the wimp chickens” means to count the number of other chickens that each chicken pecks, and then remove from the flock the chicken (or chickens) with the lowest peck count. Then repeatedly cull the flock as many times as possible. Prove that the culling method always produces an odd number of king chickens.
6. It is an unexpected fact that the culling method of finding king chickens might eliminate the chicken(s) with the highest peck count in the original flock. Construct an example of a pecking order in which this occurs.

In the remaining problems, we will say that a flock is *equitable* if it is possible to arrange all the chickens in a circle so that each one pecks the next going clockwise around the circle. On the other hand, let us say that the flock has *bullies* if it is possible to split the flock into two subflocks, each with at least one chicken, so that every chicken in the bully group pecks all the chickens in the other group.

7. Show that if a flock has bullies, then all the kings are bullies.
8. Demonstrate that if a flock is equitable, then it has no bullies.
9. Prove that if a flock has no bullies, then it is equitable. We will accomplish this in two steps. First prove that a ring of some size must exist in which each chicken pecks the next.
10. To finish the previous problem, show that if there are leftover chickens, then it is always possible to incorporate one or more of them to enlarge the ring.

Hints and Answers.

1. One efficient approach is to consider a chicken C with a maximal peck count. If another chicken C' were to peck this chicken, then by transitivity C' would also peck all the chickens that C pecks, contradicting the fact that no chicken pecked more than C . Hence C pecks all the other chickens.

2. Call the chickens C_1 through C_6 , and arrange them clockwise around a circle with

$$C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow C_4 \rightarrow C_5 \rightarrow C_6 \rightarrow C_1.$$

Then set the pecking order so that

$$C_1 \rightarrow C_3 \rightarrow C_5 \rightarrow C_1 \quad \text{and} \quad C_2 \rightarrow C_6 \rightarrow C_4 \rightarrow C_2.$$

(Note the opposite order for the second trio of chickens!) This will guarantee that all chickens are kings.

3. By the lemma, if that king is pecked, then it is pecked by another king. Hence it cannot be pecked if there are no other kings.

4. The assertion is false, although this does not become apparent until one considers relatively numerous flocks. For example, to create a counterexample with a flock of seven chickens let a king K peck field marshals F_1 , F_2 , and F_3 while K is pecked by the remaining chickens C_1 , C_2 , and C_3 . Next have the field marshals peck each other in a circle, and similarly for the other chickens. Finally, set $F_j \rightarrow C_j$ but let $C_j \rightarrow F_k$ for $j \neq k$. Then no field marshal is a king.

5. After the last cull, the remaining chickens must all have the same peck count or else it would be possible to perform another culling. In a flock of n chickens there are $\binom{n}{2} = \frac{1}{2}n(n-1)$ pecks to go around, so we need the latter expression to be evenly divisible by n . This occurs only if n is odd, when $(n-1)/2$ is an integer.

6. Take the same pecking order for a flock of seven chickens as two problems previously, except in the last step let $F_j \rightarrow C_k$ for all j and k . Then the field marshals have the highest peck count, but the king K is the only chicken to survive the culling process.

7. Clearly no chicken in the oppressed group will peck a bully, either directly or via a field marshal. Therefore any king must be in the bully group.

8. Equally clearly, given any division of the flock into two groups, the ring guaranteed by the equitable condition must eventually pass from one group to the other and then back again. Hence it is not possible for all the chickens in one group to peck all the chickens in the other.

9. Let C_1 be any chicken. Then C_1 must peck some other chicken, since otherwise we could place C_1 in one group by himself and declare the rest of the chickens to be bullies. So suppose that $C_1 \rightarrow C_2$. Then C_2 must peck some other chicken also, by the same reasoning, say $C_2 \rightarrow C_3$. Continue extending this chain until eventually we reach a point where the last chicken pecks someone earlier in the chain, at which point a ring will be formed.

10. This problem is challenging. One strategy is to argue that if there is a chicken C outside the ring which pecks at least one chicken in the ring and is also pecked by at least one chicken in the ring, then it is possible to insert C somewhere in the ring. If no such chicken exists, then everyone outside the ring either pecks the entire ring or is pecked by the entire ring. There must be at least one of each type, or else we would have bullies. Now argue that it is possible to insert these two chickens into the ring.

Chapter 8

Into the Unknown

The fourth dimension (★)

Overview. Students have always been fascinated by the notion of four dimensions, perhaps because it is a good example of a space that can be analyzed and understood mathematically, even though it cannot easily be visualized. Science fiction and modern physics alike imagine what might be possible with more than the usual three degrees of spatial freedom. Recently, the realm of four-dimensional space has even made its way into the public consciousness via the mathematical community with the advent of a (presumably valid) proof of the Poincaré conjecture. This famous assertion in topology involves a phenomenon that, until recently, was understood in n -dimensional space for every dimension except $n = 4$.

Whatever the source, the fourth dimension is usually regarded at first as mysterious and complicated. Therefore it should come as a delightful revelation to students when they discover that many familiar concepts can readily be extended to four-dimensional space. This topic is especially well-suited for younger students, perhaps at the middle school level, since it does not require much in the way of background beyond elementary algebra and geometry.

Materials. Most math circles can be conducted in a perfectly satisfactory manner without teaching aids beyond the usual pencil, paper, and chalkboard. However, this circle in particular benefits from a few extra manipulatives which provide an effective antidote to the intrinsic abstraction of four dimensions. My favorite activity involves having students build their own models of a hypercube from Zome struts. These colorful plastic pieces may be ordered online in advance at www.zometool.com or obtained at a handful of retail stores. It is also quite possible that there will already be enough parts among the students in the math circle. Creative alternatives such as thin wooden skewers and small styrofoam balls will probably do the job as well. Another nice demonstration involves a pair of scissors and several three foot lengths of rope which have been turned into loops or knots. Both of these activities are described in detail below.

Problems.

1. Why do we usually refer to a flat surface, such as a sheet of paper or a chalkboard, as a two-dimensional space?
2. How many quantities are needed to specify a single colored pixel on a standard computer monitor?
3. In order to find the distance between two points in four-dimensional space, one must first understand distance in two and three dimensions. To begin, what is the distance between the points (2006, 2007) and (2010, 2002)? In general, how do we compute distance in the plane?
4. Now figure out how far apart the “prime” points (2, 5, 11) and (3, 7, 13) are in three dimensions. Also calculate the distance from the origin (0, 0, 0) to the point $(1, \frac{1}{2}, \frac{1}{3})$.
5. Demonstrate your mastery of the fourth dimension by extending the ideas begun in the previous two problems to predict the distance between the points (1, -1, 0, -1) and (0, 1, 4, 9). Then find two points with integral coefficients that are exactly seven units apart.
6. Why is time considered a fourth dimension? Why is it really not like the fourth dimension that people usually have in mind?
7. What is the volume of a unit right isosceles 4-simplex? This is the four-dimensional polytope with vertices at (0, 0, 0, 0), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1).
8. How many “geometric components” are there in a hypercube?
9. How can one generate a cube beginning with a square? Using an analogous procedure, generate a hypercube from a cube.

Presentation Notes. It is good practice to begin math circles with a question; the natural candidate in this instance is to find out what comes to mind when hearing the phrase, “the fourth dimension.” With luck someone will suggest that time is the fourth dimension. While it is true that time is one example of a fourth dimension, space-time is not the analogue of three-dimensional Euclidean space that one might suppose, for a very fundamental reason that we will be able to explain shortly. But if nobody mentions time, there is no need to give away what’s coming up soon.

Either way, launch into a discussion of what dimension means in the first place by asking students to give their opinion as to the dimension of the space consisting of all points on the surface of the chalkboard. When the obvious answer is given, discuss why the dimension should be two. Students might appeal to the fact that there are two perpendicular directions in the plane, or that two coordinates are necessary, for example. At this level it is advisable not to put forth a technical or rigorous definition; just impress upon students that two quantities are necessary for specifying the location of a given point in the space. One way to stir up discussion is to imagine blindfolding a student near the front left of the group and asking the rest how they might direct this hapless individual to a certain seat near the back right of the room. Then ask how they would guide this person in the absence of any convenient rows and columns provided by the seats. These scenarios naturally give rise to the concept of Cartesian and polar coordinates. The precise formulation of these coordinate systems is not essential; the

point is that any “reasonable” method of describing location or displacement in the plane involves two quantities.

As an entertaining diversion, one could then consider the space of all colored pixels on a standard RGB monitor and inquire as to how many quantities one must specify to the computer in order to illuminate the desired pixel in the desired hue. The answer is five: x -coordinate, y -coordinate, red value, green value, and blue value. (A possible misconception is that a brightness level must also be given to distinguish between light green and dark green, for example. But this can be controlled by adjusting the three color parameters while keeping them in the same ratio.) In other words, we have just identified a five-dimensional space. In fact, for most dynamic applications of a monitor, such as playing your favorite movie, the time that the pixel is lighted also matters. Hence this could legitimately be construed as a six-dimensional space. No wonder it took so long to devise effective DVD-playing capability for computers!

The time has come to complement our discussion with some concrete computations, so ask students for various geometric measurements that can be made in the plane or in space. We’re looking for examples such as length or distance, area, angle measure, surface area, or volume. Mention that it is possible to compute these sorts of quantities in four-dimensional Euclidean space as well. To grasp how, we’ll first understand the process in lower dimensions, then look for a pattern that will enable us to extend our technique of calculation to four dimensions. To illustrate this strategy, begin with the concept of distance. Depending on the level of background in the audience one might need to carefully develop the distance formula in two and three dimensions or just review it more briefly. Include a nice selection of questions and allow students to explain their answers at the board. Several entertaining distance questions are given above; the answers to the first two are $\sqrt{41}$, 3, and $\frac{7}{6}$, in order. It takes only a small leap of faith to predict that the distance between points (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) in four dimensions is given by

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2}.$$

Therefore the answer to the first part of the next distance question is 11. There are many examples of two points with integer coordinates which are a distance of seven apart: $(0, 0, 0, 0)$ and $(2, 2, 4, 5)$ give one possibility.

We are now in a good position to understand why time is not “the fourth dimension” in the sense that most people expect. But first, it is certainly true that time is a fourth dimension. This can be motivated by asking students for the quantities that are typically used to pinpoint the occurrence of a crime. The location (three spatial dimensions) is certainly important, but as any fan of mystery novels knows, we cannot overlook the exact moment (one time dimension) that the crime took place. Then explain that an important step forward in the theory of relativity was the development of the Minkowski metric, which is the proper way to determine the “space-time distance,” or interval, between two points. It is given by

$$\text{interval} = \sqrt{(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2}.$$

The negatives in front of the spatial variables make all the difference. (Pun intended.) They permit two distinct points to be a “distance” of zero apart, which never occurs in Euclidean space. This will be the case for two points separated by the path of a particle travelling at the speed of light, such as a photon.

Depending on the time available one could now ask for the volume of a unit right isosceles 4-simplex. Of course, students will want to know what exactly a 4-simplex is,

and it is only sporting to tell them: it is the four-dimensional analogue of a segment, triangle, and tetrahedron. From here they may be able to tease out the answer, which would be great. If they need help, set up a chart with dimension in the left-hand column, a diagram in the next column, the quantity being measured in the third column, and the answer in the final column. For example, the second row would contain the entries ‘2,’ a picture of a right isosceles triangle with legs of length 1, ‘area,’ and ‘ $\frac{1}{2}$.’ Students should conjecture that the relevant formula in four dimensions is

$$\text{four-dimensional volume} = \frac{1}{4}(\text{volume of base})(\text{height}).$$

Hence the desired answer is $\frac{1}{24}$. In general, the volume of an n -dimensional unit right n -simplex will be $\frac{1}{n!}$.

A second optional activity which nicely complements the computational nature of our exploration thus far involves untying knots in four dimensions. You will need to prepare ahead of time by cutting several three foot lengths of rope. (String is too flimsy for this activity—look for rope at least 5 mm thick.)



Connect the ends of one rope with duct tape to create a loop, then create trefoil knots (shown at left) with the remaining lengths of rope. At the appropriate moment in the math circle, produce the loop and lay it on a table or desk. Then ask whether it is possible for a marble placed inside the loop to escape if both the marble and the rope are confined to the two-dimensional surface of the table. (No.) But if the rope is allowed to move into a third spatial

dimension then the task is easily accomplished; just lift one portion of the rope and roll the marble past, then replace the rope. Emphasize that from the point of view of the marble, which lives in the two-dimensional world of the desk, part of the rope simply vanished, leaving space for the marble to roll through. This demonstration can also be done very effectively with a smaller loop and a coin on an overhead projector.

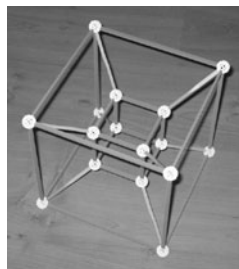
After performing the marble escape trick, pass out several of the trefoil knots and challenge students to move the rope (without cutting it, of course!) until it resembles the plain loop. When they are convinced that this is impossible, ask how the operation could be carried out in four dimensions. Let them wrestle with the problem for a while; encourage them that it can be done. At some point cut the knot where the rope was taped together (or untape it, if no scissors are available) and hold the ends a few inches apart. State that the rope is actually still a single continuous, unbroken knot; you have just moved a portion of the rope into the fourth dimension. (Remind students of the marble’s point of view from before.) They should then be able to pass the rope through the newly opened space to unknot the trefoil knot into a loop. Conclude by taping the ends back together to signify the return of the portion of rope that was hanging out in a fourth spatial dimension during the unknotting process.

Leave approximately twenty-five minutes for the final exploration. Save this activity until the end, since unveiling the Zome pieces too early in the math circle will result in distracted students who want to build nifty creations rather than concentrate on more mathematics. Begin the investigation by asking students, “How many geometric components are there in a hypercube?” Have them come up with a strategy and an answer entirely by themselves as much as possible. In particular, it is important not to define the term “geometric component.” If nobody has heard of a hypercube, then explain that it is the four-dimensional analogue of a segment, square, and cube. There are many possible ways to count the number of geometric components in the lower

dimensions. (For example, a square has 2 building blocks—points and segments, or it has 4 sides, or it is based on 4 vertices, or it contains 8 components—sides and vertices; or 9 components if you also include the interior.) Record as many as can be justified without indicating a preference. If students forget to include the interior of the square, point out that they probably included the faces of the cube in their count for the three-dimensional case, so for the sake of consistency they might want to include the single square face in the two-dimensional case. For the same reason, they may wish to include the interior of the cube for the three-dimensional case.

As long as they have been written down, it is hard to avoid noticing that the numbers 1, 3, 9, and 27 appear in successive rows. This striking pattern in turn dictates how we ought to define “geometric component.” It’s as if mathematics itself is telling us the proper definition! Then remind students that while finding neat patterns is exciting, what really makes the whole enterprise worthwhile is the process of discovering the beautiful mathematics behind the nice pattern. So then have students figure out how to generate a square from a segment, or a cube from a square. The basic idea is to draw a square, draw an identical copy of the square translated in a perpendicular direction, then join all components of the original square to the corresponding components of the copy. Students can then count the total number of components in the resulting object by adding 9 components in the original square, 9 components in the copy, and 9 components obtained by the joining process. It helps to see that joining two points gives an edge, joining two edges gives a face, and joining the two squares gives the entire cube. Hence the number of components triples with each new dimension, explaining the pattern observed.

Once you are convinced that students understand the process for using a cube to generate a hypercube (construct a cube, a copy of a cube, then join all corresponding components) they are ready to build one for themselves. Instruct students to pair up, then send one representative forward to collect Zome pieces. You will need to have on hand 12 long and 12 short blue struts, 8 medium yellow struts, and 16 white nodes for each pair of students. I would recommend simply handing them the parts and telling them to construct a hypercube without providing further directions. They will enjoy deducing how to put the pieces together. (A picture appears at right.) It is very satisfying to identify all the points, segments, squares, cubes, and hypercubes present within the model. The total number of components should come to $16 + 32 + 24 + 8 + 1 = 81$, as predicted.



To bring closure to the entire exploration, ask students whether the model in front of them is a “real” hypercube. Help them appreciate that the answer is yes in a combinatorial sense: all the components are present and accounted for. Point out that it is just as real as the picture of the three-dimensional cube on the two-dimensional blackboard. Of course, in a geometric sense the model is not real; the edges are not all perpendicular to one another and the cubes are different shapes due to the perspective of the projection. At the end of the day, be sure to have the students take their models apart slowly and carefully—the yellow struts especially are susceptible to having their tips snapped off. Have a great time with this math circle. It is a lot of fun to lead, and the students will enjoy experimenting with something they thought was inaccessible.

Further Problems.

1. Draw a picture of a cube. Keeping in mind how we used a square to generate a cube, draw a picture of a hypercube as well. (This will look different than the model you constructed using Zome.)
2. Is it possible for an ant to begin at one vertex of a cube, crawl along all the edges, and then return to its starting point without retracing any portion of its path? Is this possible on a hypercube? (Make sure that you have an accurate hypercube sketch first.)
3. How many geometric components are there in an 4-simplex? (A 4-simplex is the four-dimensional analogue of a segment, triangle, and tetrahedron.) Based on any patterns you observe, conjecture a formula for the case of an n -simplex.
4. Prove your conjecture by illustrating how an n -simplex may be built from an $(n - 1)$ -simplex. (Start with $n = 2$ and $n = 3$ first to gain intuition.)
5. Describe how to untie a trefoil knot in four-dimensional space.
6. A ball of radius r consists of the set of all points whose distance from a given point (the center of the ball) is r or less. Describe how a ball looks in one, two, or three dimensions. Then state a formula for the length, area, or volume of such a ball. Based on these expressions, what sort of formula would you expect for the volume of a four-dimensional ball of radius r ?

Hints and Answers.

1. Draw two non-overlapping, congruent cubes separated by a small distance diagonally. Then connect corresponding vertices with eight segments.
2. This is not possible on a cube, as trial and error will suggest. But it can be done without difficulty on a hypercube; use the diagram from the previous problem.
3. There are 15 components in a 4-simplex; in general there will be $2^n - 1$ components in an n -simplex.
4. The construction is carried out by plotting one additional point in the new dimension, then connecting it to every component of the $(n - 1)$ -simplex. This doubles the number of existing components and also adds 1 (the additional point). This recurrence leads to the formula stated above.
5. This process was described in the presentation notes. The problem appears here since this activity will probably be one thing too many for most circles.
6. The balls described manifest themselves as a segment of length $2r$, a circle with area πr^2 , and a sphere having volume $\frac{4}{3}\pi r^3$. It is safe to predict that the next formula should involve an r^4 term. It is considerably less obvious that there should be a π^2 factor. The rational in front is $\frac{1}{2}$, which is essentially impossible to guess. The final volume formula is $\frac{1}{2}\pi^2 r^4$.

Chapter 9

Sneaky Segments

Relatively prime integers (★ ★ ★)

Overview. A classic problem for advanced high school students, introducing ideas from algebraic number theory in an engaging yet elementary manner, involves determining the probability that two integers chosen at random are relatively prime. Really, part of the puzzle is to make sense of what it means to “choose integers at random” in the first place. Of course, the math circle environment is an ideal setting for such discussions. The development given here presents this classic problem in a new guise which provides further opportunity for dialogue and also helps to motivate the problem of relatively prime integers.

Background. No matter how this problem is presented, its resolution ultimately depends on determining the value of the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. The story of this famous problem, known as the Basel problem, is retold by William Dunham.¹ He explains that Jakob Bernoulli, writing from Basel, Switzerland in 1689, summarized the current state of mathematical knowledge of infinite series in his *Tractatus de seriebus infinitis*. This work contained many now familiar examples of geometric series, telescoping series, and common Taylor series expansions for elementary functions such as e^x , $\sin x$, and $\ln(1+x)$. Along with his own considerable contributions to the subject, Bernoulli included an admission that despite great efforts, he was unable to evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. His call for progress on this problem went unmet until after his death, when Leonhard Euler began working on the problem during the early 1730's. Initially, Euler was only able to provide vastly improved approximations to this quantity, by means of some clever manipulations with series and integrals. His subsequent demonstration in 1735 that the value of this series is exactly $\frac{\pi^2}{6}$ catapulted him to fame (within certain circles) and greatly helped to launch him on a highly successful and prolific mathematical career.

Problems. A *lattice point* is a point in the Cartesian plane whose coordinates are both integers, such as $(1, 0)$ or $(-4, 7)$. Let us define a *sneaky segment* to be

¹*Euler, The Master of Us All*, William Dunham, Mathematical Association of America, pp. 39–48.

a segment extending from one lattice point to another without passing through any other lattice points along the way. For example, the segment with endpoints $(1, 0)$ and $(-4, 7)$ is sneaky.

1. How many lattice points lie on the circumference of the circle with radius 5 and center at the origin?
2. Consider the sixteen lattice points whose x and y -coordinates are each a positive integer from 1 to 4, inclusive. If we join each of these points to the origin $(0, 0)$, how many of the resulting sixteen segments are sneaky?
3. Show that if m and n are integers not both zero, then the segment with endpoints $(0, 0)$ and (m, n) is sneaky if and only if m and n are relatively prime.
4. Why does it not make sense mathematically to choose an integer at random?
5. Despite the foregoing difficulty, why can we talk about the probability that an integer “chosen at random” is divisible by 10?
6. Using your idea from the previous problem, compute the probability that a “random integer” is not divisible by 7.
7. Determine the probability that a randomly chosen integer is not divisible by 7 but is divisible by 3.
8. If two integers are chosen at random, what is the probability that they are both divisible by 5?
9. On the other hand, given two random integers, find the probability that they do not share a factor of 5. (It is fine for one of them to be divisible by 5, as long as the other is not.)
10. What is the probability that two integers “chosen at random” are relatively prime? Write your answer as an infinite product, with one factor for each prime.
11. Demonstrate that

$$\frac{1}{1 - \left(\frac{1}{2}\right)^2} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{16^2} + \cdots.$$

12. Find a degree three polynomial $f(x)$ which has zeros at $x = -2$, -4 , and 5 , and which further satisfies $f(0) = 1$.

Presentation Notes. To begin, make sure that everyone is familiar with the Cartesian plane and the concept of coordinates, then ask if anyone has ever heard of a lattice point. A knowledgeable individual (or yourself, if volunteers are in short supply) can explain that lattice points have integer coordinates. Draw a grid of lattice points on the board (better yet, come prepared with an overhead slide) and mention that one can join any pair of lattice points with a segment. Such a segment might run over another lattice point, as is the case for the segment joining $(1, 2)$ with $(5, 4)$. However, if we’re careful, we can avoid all the other lattice points. In this case we say that we have a sneaky segment. Ask students to find examples of sneaky segments and include a nice variety of them on your grid of lattice points.

By now students may have already pointed out that a segment is sneaky if its ‘rise’ and ‘run’ are relatively prime integers, or made a similar observation. (Otherwise, ask

leading questions such as “Which of the segments joining $(19, 14)$, $(20, 14)$, and $(21, 14)$ to the origin are sneaky?” Then inquire whether there is a quick way to predict the answer without drawing a detailed picture.) Facilitate the formulation of a precise number theoretic test for determining sneakiness, such as “The segment joining (a, b) and (c, d) is sneaky if and only if the integers $(a - c)$ and $(b - d)$ are relatively prime.” It would probably only derail the train of thought to include a rigorous proof of this fact at this point; an informal argument given by a student explaining what goes wrong when there is a common factor should suffice.

The burning question is, “If two lattice points are chosen at random, what is the probability that the segment which they define is sneaky?” Point out that there is something highly suspect in the phrasing of the question and check whether anyone caught it. Prompt them if necessary by asking for the probability that a person chosen at random from the room will be left-handed. Students should catch on to the fact that their standard definitions of probability break down for an infinitely large sample set. Now that they appreciate the need for an alternate approach, ask how one might compute this probability. There are several possible methods, including empirical approximation, a limiting process, and the technique of computing the probability “at each of the primes” separately. (We will make sense of this phrase shortly.)

Finding an empirical approximation is entertaining if there are enough students with scientific calculators in the room. Have them enter `100*rand` and copy down the integer part of the result to obtain a random integer from 0 to 99. By repeatedly pressing the **[Enter]** key they should be able to generate a list of twenty integers in this range. By grouping them into ten pairs, students can then check whether the numbers in each pair are relatively prime. This will simulate choosing ten segments and counting how many of them are sneaky. Tally the results (the more the better, naturally) to obtain a rough approximation of the actual probability. Your answer should be in the neighborhood of 60% of the time. If desired, you could also have a computer conduct a more thorough experiment to provide another estimate.

If it has not come up already, explain that we may assume without loss of generality that the initial endpoint of our segment is the origin. In other words, we could choose two lattice points and check whether they define a sneaky segment. Alternatively, we could choose only a single lattice point, connect it to the origin, then ask whether the resulting segment is sneaky. These two procedures are essentially equivalent, hence will yield the same probability. This simplifying assumption was already implicit in the simulation above. Finally, guide students to an intuitive understanding of what would be involved in extending this process in the quest for an exact answer. Thus one might define $p(n)$ to be the probability that two random integers from 0 to n are relatively prime, then declare the limiting value of $p(n)$ as n approaches infinity to be the probability we seek. Highlight some of the issues involved in this technique, such as the issue of whether the limit exists, or is dependent on the route to infinity. (I.e. what if we looked at expanding circles of points rather than squares of points?) The goal is to acquaint them briefly with a few notions from analysis.

Another method for circumventing the difficulty of an infinite sample set is to find an answer to our central problem “at each prime” and then combine all the results. To motivate this approach, ask the series of four questions related to divisibility appearing above. (The answers are $\frac{6}{7}$, $\frac{2}{7}$, $\frac{1}{25}$, and $\frac{24}{25}$.) Emphasize that the beauty of this approach is that where divisibility by 7 is concerned, there are only seven kinds of integers: those which are evenly divisible by 7, those that leave a remainder of 1 when divided by 7, those that leave a remainder of 2, and so on. Hence our familiar rules of probability apply. Also point out that divisibility by 3 and by 7 are independent, in the sense

that every possible combination of remainders occur exactly once in any string of 21 consecutive integers. (This is the Chinese Remainder Theorem in an elementary form.) Therefore we can multiply probabilities as hoped.

The stage is now set for tackling the main computation. See if students can write down an expression for the probability that a segment chosen at random is sneaky. If necessary, remind them that the preceding discussion implies that this question is equivalent to finding the probability that two integers chosen at random are relatively prime. Further steer them, as necessary, to realize that the condition of being relatively prime can be checked at each prime separately. In other words, the two integers cannot have a common factor of 2, and they cannot have a common factor of 3, and so on. The above exercises then dictate that this probability may be written as

$$p = \left(1 - \left(\frac{1}{2}\right)^2\right) \left(1 - \left(\frac{1}{3}\right)^2\right) \left(1 - \left(\frac{1}{5}\right)^2\right) \left(1 - \left(\frac{1}{7}\right)^2\right) \cdots$$

It is important to write the factors in this form rather than as $\frac{3}{4} \cdot \frac{8}{9} \cdot \frac{24}{25} \cdots$ to pave the way for what comes next. You could also spend a moment discussing the meaning of an infinite product such as this. At the very least, note that a limiting process creeps into the process even with a number theoretic approach.

Now announce that the situation is going to progress from bad to worse: rather than handling the expression for p directly, you will examine the reciprocal $\frac{1}{p}$ instead. Review geometric series and laws of exponents to the extent necessary so that students are comfortable writing

$$\frac{1}{1 - \left(\frac{1}{2}\right)^2} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{16^2} + \cdots$$

Perform a similar substitution for each of the factors of p . It can be quite dramatic to see this infinite product of infinite series stretching across the top of the blackboard; include at least four series, out to the $(1 + \frac{1}{7^2} + \frac{1}{49^2} + \cdots)$ term. And now for the punch line: ask students to multiply out this product of series. Your request may elicit shock and disbelief; all the better. Encourage them to name even a single term in the overall result. Some brave soul may suggest that it begins with $1 + \cdots$, otherwise point out this fact and let them take it from there. It will gradually dawn on them that the reciprocal of every perfect square appears in the sum exactly once. Seeing this realization spread across the room is a beautiful thing. To reinforce their observation, point out that this phenomenon occurs because every positive integer factors uniquely into a product of powers of distinct primes. You might also caution the group that these sorts of manipulations do require justification, but we'll leave the details to the analysts for now and enjoy the number theory.

The time has come for relating the story of the Basel problem mentioned in the introduction above, but hold off on revealing the precise value of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. After relating the tale of Euler's triumph, outline his method for calculating this value. (The approach given here is the same in spirit to Euler's, but differs slightly in presentation.) Begin by displaying the curious, but seemingly unrelated function

$$g(x) = \frac{\sin \pi x}{\pi x}.$$

Our goal will be to derive two different series expansions for $g(x)$. On the one hand, Euler was quite familiar with the Taylor series expansion for $\sin x$. Ask if anybody in

the room knows it off the top of their head:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

where $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ and similarly. Have students then perform the algebra necessary to get expansions for $\frac{\sin x}{x}$ and then $\frac{\sin \pi x}{\pi x}$, obtaining

$$\frac{\sin \pi x}{\pi x} = 1 - \frac{\pi^2 x^2}{6} + \frac{\pi^4 x^4}{120} - \frac{\pi^6 x^6}{5040} + \cdots.$$

The second formula comes about by identifying the zeros of $g(x)$. The group should agree that these occur at $x = \pm 1, \pm 2, \pm 3$, and so forth. There may be some debate as to what happens at $x = 0$; if they need direction remind them of the series expansion just obtained, which suggests that $g(0) = 1$. Hence $x = 0$ is not a zero of $g(x)$. Remind students of how to reconstruct a function from its zeros by giving them an exercise for practice, “Let $f(x)$ be a degree three polynomial with zeros at $x = -2, -4$, and 5 . Find a formula for $f(x)$.” They should decide that

$$f(x) = C(x+2)(x+4)(x-5),$$

where C is an arbitrary non-zero constant. If they forget the constant, announce that their formula works, but you were thinking of a different one—can they figure out why there is more than one answer? Next state that you have a formula in mind for which $f(0) = 1$. Guide them to discover the nice symmetric formula for $f(x)$,

$$f(x) = \left(1 + \frac{x}{2}\right) \left(1 + \frac{x}{4}\right) \left(1 - \frac{x}{5}\right).$$

Students will now be able to conjecture a formula for $g(x)$ based on its zeros and the fact that $g(0) = 1$. Suggest that they combine pairs of factors to obtain the more compact expression

$$g(x) = \left(1 - \frac{x^2}{1^2}\right) \left(1 - \frac{x^2}{2^2}\right) \left(1 - \frac{x^2}{3^2}\right) \left(1 - \frac{x^2}{4^2}\right) \cdots.$$

Explain that this formula turns out to be valid, although our derivation required a leap of faith at the end. Point out that the function $e^x g(x)$ has precisely the same zeros and also evaluates to 1 at $x = 0$, but will obviously not be given by the above product expansion. Accepting this formula for $g(x)$ as valid, ask students how we can capitalize on the fact that we now have two different expansions for $g(x)$. Prod them to figure out where the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ will appear. With some guidance, they will focus on the coefficient of the x^2 term in each expansion. Equating coefficients yields

$$-\frac{\pi^2}{6} = -\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Therefore the mysterious value of our infinite series is $\frac{\pi^2}{6}$, resolving the Basel problem. Furthermore, the probability we are seeking is the reciprocal, or $\frac{6}{\pi^2} \approx .608$. It is satisfying to compare this figure to our earlier approximations and discover that they are (usually) in surprisingly good agreement.

Further Problems.

1. Consider the twenty-five lattice points whose x and y -coordinates are each positive integers from 1 to 5, inclusive. If two of these lattice points are chosen at random, what is the probability that they define a sneaky segment?
2. Define $p(n)$ to be the probability that choosing two points at random inside the circle with radius n centered at the origin will yield a sneaky segment. What would you hypothesize should be true of the values of $p(n)$ as n goes to infinity?
3. Let us say a segment is “almost sneaky” if it passes through exactly one other lattice point. Prove that the probability that two lattice points chosen at random in the plane determine an almost sneaky segment is $\frac{3}{2\pi^2}$.
4. Suppose we work with a hexagonal lattice instead, so that the points form equilateral triangles instead of squares. Find the probability that two lattice points chosen at random create a sneaky segment. (HINT: there is no need to start from scratch; just relate this problem to our original question.)
5. The product formula for $\sin(\pi x)$, rewritten as

$$\frac{\sin(\pi x)}{\pi x} = \left(1 - \frac{x}{1}\right) \left(1 + \frac{x}{1}\right) \left(1 - \frac{x}{2}\right) \left(1 + \frac{x}{2}\right) \cdots,$$

gives rise to some pretty weird formulas. For example, substitute $x = \frac{1}{2}$ to confirm a formula for π due to John Wallis: $\frac{2}{\pi} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdots$. What other unusual formulas can you create by using other strategic values for x ?

6. There is a clever method of finding the exact value of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$. Try replacing x by ix in our expressions for $g(x)$, then strategically combine the new identities with the original ones.
7. You are now ready for the following variation on the sneaky segments problem: what is the probability that two points chosen at random from a four-dimensional square lattice determine a sneaky segment? You can think of these lattice points as having integer coordinates (a, b, c, d) .

Hints and Answers.

1. There are $\binom{25}{2} = 300$ segments; carefully count how many of them are *not* sneaky. Remarkably, the answer simplifies to exactly $\frac{2}{3}$.
2. We expect $\lim_{n \rightarrow \infty} p(n)$ to be $\frac{6}{\pi^2}$, just as before.
3. Equivalently, we want randomly chosen integers m and n to have exactly one factor of 2 in common. This changes the initial factor of $\frac{3}{4}$ in our original expression into $\frac{3}{16}$.
4. The problems are equivalent, since an affine transformation converts a square lattice into a hexagonal lattice. So the answer is still $\frac{6}{\pi^2}$.
5. For example, $x = \frac{1}{6}$ yields $\frac{3}{\pi} = \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdots$.
6. Multiply the product and series expansions for $g(x)$ and $g(ix)$, then compare the coefficients of x^4 to discover that the series evaluates to exactly $\frac{\pi^4}{90}$.
7. An identical chain of reasoning as in the two-dimensional case gives $\frac{90}{\pi^4} \approx .924$.

Chapter 10

Heads or Tails

Probability and Expected Value (★★)

Overview. The concept of uncertainty holds an intrinsic fascination; games of chance have been popular throughout the ages. The desire to analyze certain dice games motivated Pascal, in conjunction with Fermat, to develop much of the elementary theory of probability during 1654. The fact that relatively innocuous questions can lead to substantial mathematics makes probability a delightful topic for a math circle. Nothing beyond a dollar's worth of pennies is needed for the activities described in the following paragraphs, but the problems they generate will keep students occupied well beyond the meeting.

Problems. Suppose that a fair coin is tossed ten times. We may record the outcome of such an event as a string of ten H 's and T 's, with the understanding that the probability of an H or T occurring at any particular position is $\frac{1}{2}$. Most of the questions below address this situation.

1. If this experiment is conducted twenty times, then in how many cases would one expect the sequence of ten H 's and T 's to begin with a T ?
2. How many outcomes are possible when flipping a coin ten times?
3. In how many of these outcomes will there be exactly three heads? Consequently, what is the probability of obtaining three heads?
4. If the experiment is conducted twenty times, then in how many cases would one expect to obtain exactly three heads?
5. On the other hand, in how many of the twenty trials do we expect to obtain exactly six heads?
6. Consider all the possible outcomes from tossing a fair coin ten times. In how many cases will the first tail occur on the very first flip? In how many cases will it occur on the second flip?
7. Make a table listing the number of outcomes in which the first occurrence of a tail is on the k^{th} flip, for each k in the range $1 \leq k \leq 10$.
8. Based on your table, calculate the average number of flips required to obtain the first tail.

9. Suppose we were to flip a coin repeatedly until it came up tails. What is the expected number of flips needed to accomplish this task?
10. There is a subtle difference in the previous two questions. What is it?
11. Given an unfair coin, how could we empirically determine the probability (presumably not equal to $\frac{1}{2}$) that the coin comes up heads?
12. What does it mean to say that an event will occur with probability $\frac{2}{3}$? What about with probability $(\sqrt{2} - 1)$?

Presentation Notes. Human beings are notoriously unreliable at mimicking random behavior, such as the outcomes of a succession of coin tosses. Therefore the following activity makes for an entertaining way to begin a math circle on probability. To begin, have each student pretend that they are a coin. Then instruct the group to flip themselves ten times and record their outcomes across the top of a sheet of paper. Say as little as possible to influence how they conduct this experiment; for example, avoid the phrase “heads or tails” so as not to suggest that heads come before tails. The experiment is still effective even if there are relatively few students in the room and the deviation from true random behavior is more pronounced if each student only records one sequence of outcomes. Then distribute pennies to the students (or quarters which they can keep, if you want to achieve instant popularity) and have them list the outcomes of ten actual flips of the coin in a row on their sheet of paper below the previous sequence of heads and tails.

Once the data has been recorded, have students suggest measurements that could be made on a sequence of ten coin toss outcomes. In other words, given a string of ten H 's and T 's, what features of the sequence can be counted? Possibilities include

- the number of heads (or tails) that appear in the list,
- the number of times the sequence switches from a string of heads to a string of tails or vice-versa (or equivalently, the number of blocks of all heads or all tails),
- the length of the longest string of all heads or all tails,
- the position of the first tail to appear in the list,
- or the value of the first coin toss.

Be sure to suggest the latter item if nobody else does; we all know that roughly half the sequences should begin with a tail, but the student-generated outcomes will invariably be skewed towards an initial head. For each proposal, have students perform the measurement on both sets of outcomes (their own attempt at random behavior as well as the results of their actual coin tosses). Then tally the results for the each type of data separately; the made-up outcomes in one column and the true coin flips in an adjacent column. Let students point out any qualitative differences that they observe.

Once the group appreciates how truly random behavior deviates from our own perception of randomness, it is natural to wonder what sorts of results one would expect from a mathematical point of view. For example, consider the matter of how many heads might appear in a sequence of ten coin tosses. Ask students for the various possibilities (anywhere from zero to ten) and for the fraction of the time we would expect each, mathematically. Depending on the level of the group and your own interest, this portion of the circle could range from a brief five-minute review of computing binomial coefficients to a much more in-depth presentation. Regardless, frame the discussion by explaining that there are $2^{10} = 1024$ equally likely outcomes

for ten flips of a coin; we are interested in the fraction of these outcomes featuring a certain number of heads. The rest is an exercise in counting. Thus there will be

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

instances in which there are three heads, which should occur $\frac{120}{1024} \approx 11.7\%$ of the time. If there are twenty students in the room, this translates to an expected $20 \cdot \frac{120}{1024} \approx 2.34$ sequences containing exactly three heads. The random data will probably reflect this figure, the student-generated outcomes will certainly not! The remaining possibilities may be computed similarly and displayed in a third column next to the first two.

We include a brief review of binomial coefficients here in case it is helpful. The number of ways to choose k objects from among a collection of n distinct objects is denoted $\binom{n}{k}$, often read “ n choose k .” This quantity may be computed using the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1}.$$

The case $n = 10$ and $k = 3$ is illustrated above. To count the number of ten-flip outcomes which contain three heads, we recast the question as follows. Imagine drawing ten blanks in a row, each of which will contain a H or T . We must choose three of the blanks to fill with H 's. This can be done in $\binom{10}{3}$ ways. Similarly, the number of sequences containing exactly six heads will be $\binom{10}{6} = 210$, which will occur $\frac{210}{1024} \approx 20.5\%$ of the time. The complete table is compiled below.

heads	# seq	% time
0	1	0.1
1	10	1.0
2	45	4.4
3	120	11.7
4	210	20.5
5	252	24.6
6	210	20.5
7	120	11.7
8	45	4.4
9	10	1.0
10	1	0.1

Some of the items in the list of possible measurements above are rather non-trivial to compute and might be better left as problems for students to pursue on their own. (The length of the longest block would probably fall into this category.) Of course, student interest and your own expertise and judgment should dictate the flow of the discussion as much as anything. There is no set curriculum to be covered at a math circle! But stalling out on a problem that turns out to be more difficult than expected does tend to kill the momentum, so there is a balance to be struck. The computation for the length of the longest block, on average, appears in the hints and answers section.

For most groups, the most fruitful direction of inquiry at this point will be to consider the position of the first tail. You have probably already noticed that the students tended to place it at the second or third spot, whereas the distribution is more spread out in general. Nevertheless, tell students that perhaps they can redeem

themselves by demonstrating that on average their placement of the first tail was about the same as the mathematically expected value. For starters, have them compute their own average as well as that of the actual coin tosses. (If they are not sure what to do, have them come up with a strategy as a group. In the end they should determine the position of the first tail in each sequence and then average the resulting numbers.) Then ask them how to determine the mathematically precise expected value for the position of the first tail. Guide them as needed to consider all 1024 equally likely sequences of heads and tails. In 512 of them a tail will come first, while in 256 of them there will be a head followed by a tail, and so on. There is a question as to how to handle the case of all heads—point out that this anomaly suggests that perhaps there is a cleaner way of setting up the question! For the time being pick a number (perhaps 11 or 12, they can vote) with the intention of settling the matter shortly. The average is then

$$\frac{1}{1024}(1 \cdot 512 + 2 \cdot 256 + 3 \cdot 128 + 4 \cdot 64 + \cdots + 10 \cdot 1 + ? \cdot 1) \approx 2.$$

The transition to a general, more powerful method for computing expected value comes by realizing that this expression may be rewritten as

$$1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots + 10 \cdot \frac{1}{1024} + ?$$

In other words, it appears that if we list all the possible values of the quantity we are measuring (the position of the first tail, in our case) and compute the probability of obtaining each possible value (namely $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$), then the average or expected value may be found by multiplying each value by its probability and summing the results.

The next step is to ask how to avoid the anomaly observed above. Hopefully a student will propose that we rephrase the original question as, “On average, how many times must one flip a fair coin before a tail comes up?” without limiting the number of tosses to ten. The expected value becomes

$$E = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots$$

which is a neater expression to work with mathematically, despite the fact that we have now introduced an infinite series to replace the finite sums we were working with earlier. It is a nice exercise to let students evaluate this series; perhaps one of them is familiar with the standard technique of writing

$$\begin{aligned} 2E - E &= \left(\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots \right) - \left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots \right) \\ &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 2. \end{aligned}$$

Hence the precise expected value for the appearance of the first tail is after two flips.

There are several slick ways to establish this expected value without dealing with infinite series, though. It would be worthwhile to present one or more of them at this point. For instance, let the expected value be E , and imagine flipping the coin for the first time. Half the time a tail will turn up, which leads us to write $E = 1 \cdot \frac{1}{2} + \underline{\hspace{1cm}}$. But the other half of the time we are right back where we started; in other words, on average we will have to make an additional E flips before obtaining our first tail, for a

total of $E + 1$ flips in all. Hence we can complete the equation as $E = 1 \cdot \frac{1}{2} + (E + 1) \cdot \frac{1}{2}$, which quickly implies that $E = 2$.

Depending on the time available and the level of the group, this may be plenty for the day. However, there are many interesting directions in which one could go from here. For starters, we have tacitly assumed throughout that we are using fair coins; i.e. that the probability of obtaining heads is exactly $\frac{1}{2}$. To prompt students to think more deeply about what this means, ask them how they could ascertain the probability of obtaining heads given an unfair coin. It is a good exercise to formulate a precise description of their strategy. Thus one should repeatedly toss the coin and record the total number of heads $H(n)$ that appear after n flips. Then define the probability p of obtaining heads as

$$p = \lim_{n \rightarrow \infty} \frac{H(n)}{n}.$$

Mention that the fact that this limit exists (with probability one) is a non-trivial result implied by the Central Limit Theorem. One could follow up on this discussion by asking what it means for an event to occur with probability $\frac{2}{3}$. Emphasize that this does not necessarily mean that there are three equally likely outcomes, two of which are desirable. To illustrate this point, you could then have students discuss what it means for an event to occur with probability $\sqrt{2} - 1$.

There are a multitude of other nice problems based on successive tosses of a coin which would provide a stimulating finish to this math circle. Some of these are outlined in the further problems below, or you can insert your own favorite exploration. For instance, it is a real eye-opener to have students guess how many flips it will take, on average, before the number of heads exceeds the number of tails. It seems clear that it won't take long, since there is a fifty percent chance of success on the very first flip. But as the group begins to conduct the actual experiment and record the number of flips needed, they will gradually realize that the average only gets higher the longer they try. (Make sure that each person conducts the experiment once and records a number before letting the group as a whole move on to conduct another trial!) A brief explanation of why this average diverges to infinity is outlined below.

Further Problems.

1. Suppose that a fair coin is flipped ten times. The results of the tosses can be grouped into alternating blocks of all heads and all tails. Compute the probability that there are exactly n blocks, for each possible value of n .
2. Based on the previous calculations, determine the expected number of blocks.
3. Now determine the probability that the length of the longest block is l , for each value of l in the range $1 \leq l \leq 10$.
4. Using these computations, find the expected length of the longest block.
5. Alice tosses a fair coin nine times, while Bob tosses a coin ten times. Bob wins if he obtains more heads; Alice prevails if she obtains the same number or more heads than Bob. What are Bob's chances of winning?
6. Alice and Bob take part in a new game. Bob chooses the sequence of three outcomes THH . Alice must now choose a different sequence of three outcomes. A coin is then tossed repeatedly; the winner is the person whose sequence comes up first. What sequence should Alice choose to maximize her chances of winning, and what is this optimal probability?

7. How many times do we expect to flip a coin before the second tail appears?
8. A fair coin is tossed repeatedly. On average, how many flips will take place before the number of heads exceeds the number of tails?

Hints and Answers.

1. To create n blocks, place $(n - 1)$ “dividers” into the nine spaces between coins in a row of ten coins, which can be done in $\binom{9}{n-1}$ ways. There are then two ways for the blocks to alternate between heads and tails, so dividing by 512 gives the probability.
2. Using the previous calculation, the average number of blocks comes to exactly $5\frac{1}{2}$. This makes sense, because the first coin starts a block, and on average, each successive coin starts a new block exactly half the time.
3. The number of sequences for which the longest block has length l is 2, 176, 370, 254, 126, 56, 24, 10, 4, and 2 as l ranges from 1 to 10. These numbers are tedious to compute; a computer was used to create this list. Dividing each figure by 1024 gives the corresponding probabilities.
4. Based on the above figures, the average length of the longest block is about 3.66.
5. Remarkably, this is a fair game, meaning that each player has a $\frac{1}{2}$ probability of winning. To see why, show that for any particular sequence of coin tosses for Alice and Bob, reversing the heads and tails in each sequence will always result in exchanging the winner and loser.
6. Alice should choose *TTH*. The game is slightly tricky to analyze, but in this case Alice will win with probability $\frac{2}{3}$.
7. One could employ the technique described above, letting E be the expected value and then arguing that $E = \frac{1}{2}(2 + 1) + \frac{1}{2}(E + 1)$, giving $E = 4$. More simply, we already know that it will take two flips on average to obtain the first head. By the same reasoning, it will take two more flips to reach the second tail.
8. Let E be the expected number of flips until the number of heads exceeds the number of tails by exactly one. (Assuming this quantity is finite.) Then the expected number of flips until we have two more heads than tails will be $2E$. But considering the possible outcomes of the first flip leads to $E = \frac{1}{2}(1) + \frac{1}{2}(2E + 1)$, or $0 = 1$, which is not possible. Hence E cannot be finite. Or put another way, the only “number” which satisfies $E = E + 1$ is “ $E = \infty$.”

Chapter 11

Circling the Square

Perfect Squares and Picture Proofs (★)

Overview. The seemingly innocuous procedure of multiplying a number by itself quickly leads to fertile mathematical ground, well-trod by great number theorists including Pythagoras, Fibonacci, Fermat, Euler, and Gauss, to name only a few of the best known. Many marvelous results on squares have been established; the theory of quadratic reciprocity being one of the most elegant examples. However, despite being a subject of great interest among mathematicians for centuries, there are still problems that can be simply stated but which have yet to be resolved. For instance, there is the question of whether infinitely many primes appear in the list $2, 5, 10, 17, 26, 37, \dots$ of positive integers which are one greater than a perfect square.

Its very name reflects the fact that a square number has a geometric interpretation; namely, the area of a square or the number of dots appearing in a square array. Therefore it should come as no surprise that many elementary facts about square numbers may be shown by way of a strategically constructed diagram. Such demonstrations have been traditionally dubbed ‘proofs without words,’ which we will shorten to simply ‘picture proofs’ for this presentation. Students will have an engaging time searching for the right diagram to illustrate facts about squares numbers, and this theme provides a neat thread which ties together the various topics covered in this math circle.

Problems.

1. Determine the value of $1 + 2 + 3 + \dots + 2005 + 2006 + 2005 + \dots + 2 + 1$.
2. Choose any positive integer and compute its square. Then add both your original number and the next higher integer to this square. The result will be the next perfect square! Use a picture to explain why this trick works.
3. Compute the values of 35^2 , 45^2 , 55^2 , and 65^2 . What patterns do you notice in the last two digits of the answers? In the first two digits?
4. What happens if you add up the first four odd numbers? In other words, compute $1 + 3 + 5 + 7$. What happens if you sum the first seven odd numbers? Create pictures to explain why this happens.

5. One of the numbers 212522, 213444, or 214369 is not a perfect square. Identify which one without performing any calculations.
6. Cut a 29×29 square out of one corner of a 71×71 square. Dissect the remaining L-shaped figure into two pieces that can be reassembled into a rectangle. What does this tell you about the value of $71^2 - 29^2$?
7. Compute the value of $20^2 + 19^2 + \cdots + 11^2 - 10^2 - 9^2 - \cdots - 1^2$ without using a calculator.
8. Find the prime factorizations of 3599 and 2491.
9. Can a positive perfect square be equal to exactly twice another perfect square? Either find an example or explain why it could never occur.
10. It is possible for twice a perfect square to differ from another perfect square by exactly one. For example, $2 \cdot 5^2 = 50$, which is just above $7^2 = 49$. Find at least three other examples of this sort of phenomenon.
11. List your examples from the previous problem in increasing order. Then detect a pattern which will allow you to generate more examples, and check to see if your new examples work.

Presentation Notes. Certain problems are especially well-suited for use in math circles because they are intriguing yet accessible, lend themselves to a wide variety of solutions, and illustrate sound problem-solving principles. The following is such an example, with the added bonus that it neatly introduces the topics for the day. The problem is to determine the value of the sum

$$1 + 2 + 3 + \cdots + 2004 + 2005 + 2006 + 2005 + 2004 + \cdots + 3 + 2 + 1.$$

Of course, you should replace the central number 2006 with the current year or your own favorite large positive integer. (An even number is preferable.) Have students come up with as many different methods of solution as possible. A few common approaches are outlined below.

- A common technique involves pairing numbers in some fashion, such as $1 + 2005$, $2 + 2004$, $3 + 2003$, $\dots$ on the left half of the sum, and again on the right half. In our case they will need to handle the extra 1003's left over in the middle, presumably by pairing them also. When the dust settles, there are 2006 values of 2006 to add, for a total of 2006^2 . Then press students to find a cleaner pairing scheme. Lead them to discover that they could also match the initial 1 with the second 2005, then the 2 with the second 2004, and so on. It is much clearer now that the pairing process results in 2006 values of 2006 resulting.
- Another closely related approach is to utilize the formula for the sum of the first n positive integers, $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$. Then apply it once with $n = 2006$ and again with $n = 2005$ to cover the entire sum. This method has the advantage of quickly providing an answer, especially if one has a calculator handy. The formula just stated is a good tool to know, but when applied in our situation it suffers from the drawback that it is not apparent that the answer it provides is actually a perfect square.

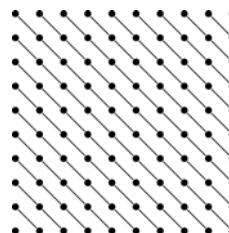
- Most calculators will evaluate sums like these if asked nicely. The pros and cons of the previous approach are simply accentuated: numerical answers can be obtained quickly and accurately at the expense of understanding or even appreciating the mathematics behind them. (Of course, calculators are terrific tools, when implemented properly.) While on the topic of arithmetic, ask students if they know of a quick way to compute 2006^2 . Walk them through the expansion

$$2006^2 = (2000 + 6)(2000 + 6) = 2000^2 + 6 \cdot 2000 + 6 \cdot 2000 + 6^2 = 4,024,036,$$

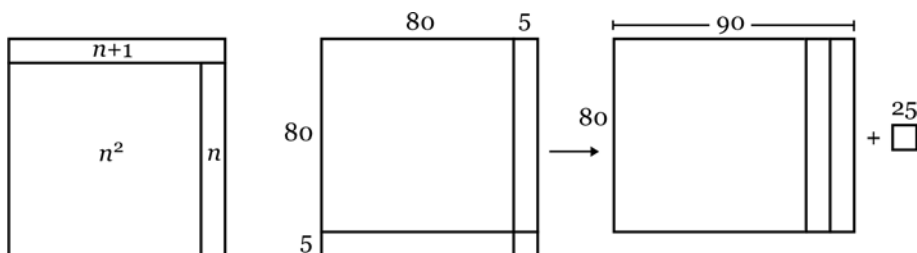
or have a volunteer present the calculation at the board.

- If nobody thinks to tackle a simpler version of this problem, ask students how they might have guessed the correct answer without working on the given sum directly. Explain that one of the most effective, widely used methods for solving difficult problems is to formulate and solve simpler versions. Have students practice this method on the given problem, for example by computing $1 + 2 + 3 + 2 + 1 = 9$. The fact that every answer is the square of the central number quickly becomes obvious, allowing one to conjecture that the above sum is 2006^2 .
- Finally, suggest that because the answer came out so neatly, there is probably an equally nice explanation for it. Ask how one could represent a square number, such as 100, geometrically. (As a 10×10 square grid of dots. Use dots rather than area for now.) Then ask students if they can find the quantities 1, 2, ..., 9, 10, 9, ..., 2, 1 visually within the array of dots. Eventually someone will notice that these are the numbers of dots along the diagonals, as shown below. It is now immediately obvious that the sum must be 10^2 without any computation! In the same way, the original sum above will equal 2006^2 . The geometric picture proof is the most elegant.

Now challenge the group to rattle off as many perfect squares as they can from memory, beginning with $0^2 = 0$. (No calculators or pencil and paper allowed!) Create a list of squares across the top of the board for later reference; have the students compute the values up to at least $20^2 = 400$ if they don't know them by heart. Then ask students for any trivia, facts, or theorems which they already know about perfect squares. It is almost always a good idea to give the audience an opportunity to share their collective knowledge on a topic; it helps one to gauge the level of the group and causes students to become more invested in the proceedings. (This brainstorming works best if contributions are kept short and are generated by many different students. Thus it is probably not a good idea for students to come to the board at this stage.) Some of the many possible avenues of exploration are described below. Pick and choose among them to fill out the middle portion of the math circle, but leave around half an hour for the final exploration.



Students often point out that adding n and $n + 1$ to n^2 yields $(n + 1)^2$. This is usually presented by way of example: “To find the next perfect square after 14^2 , just add 14 and 15.” Sure enough, $196 + 14 + 15 = 225 = 15^2$. Then verify this identity algebraically, for those students familiar with algebra. However, a more pleasing approach, which can be appreciated by everyone, is finding a picture proof of this fact. Have students search for such an explanation and then pick a volunteer to present the proof at the board. A sample diagram is shown below at left.



There is a clever shortcut for calculating certain squares that may have been mentioned while creating the list of squares earlier. To compute the square of a positive integer ending with a 5, remove the 5, multiply the remaining number by the next higher integer, then append the digits 25 to the result. For example, to find 85^2 we compute $8 \cdot 9 = 72$, so the answer is 7225. As before, there is an algebraic explanation for the identity:

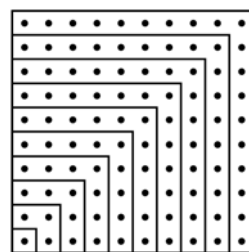
$$(10n + 5)^2 = 100n^2 + 100n + 25 = 100[n(n + 1)] + 25.$$

Unless students request it, though, forego the algebra in favor of finding a convincing picture. Some cutting and pasting is required this time, as shown to the right above.

Another elementary result on square numbers is that the sum of the first n odd numbers gives n^2 . For example, $1 + 3 + 5 + 7 = 16 = 4^2$. Then challenge students to compute the value of

$$1 + 3 + 5 + \cdots + 2005 + 2007$$

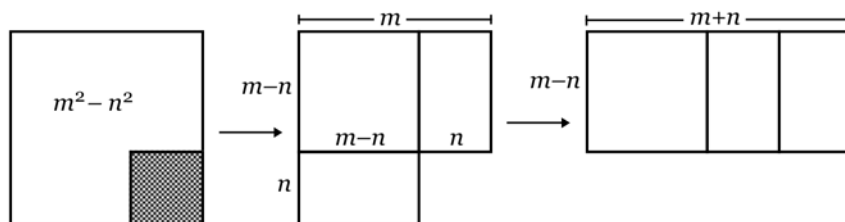
without a calculator. The total comes to $1004^2 = 1008016$. Of course, they should find a picture proof of this fact; the standard illustration is given at right for the $n = 10$ case.



Most students are aware that perfect squares can only end in certain digits, even if they don't remember to mention this fact while they are brainstorming. You can jog their memories by declaring that two of the following three numbers are perfect squares: 212522, 213444, and 214369. Their job is to spot the imposter. (The first number is not a square.) If they need a hint, tell them that one of the digits will give the imposter away. Then note that the final digits in the list of perfect squares repeats every ten numbers in the pattern 0, 1, 4, 9, 6, 5, 6, 9, 4, 1, ...; which makes sense, given the standard algorithm for multiplying numbers by hand. Hence no perfect squares end with 2, 3, 7, or 8.

Someone may also be familiar with the formula for the sum of the first n squares, which states that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$. This is a handy formula to know, although it is not immediately obvious how to prove it. Mathematicians are confident that Archimedes knew of this formula, and have hypothesized various picture proofs that he may have used to demonstrate it.

There is a fair chance that somebody is familiar with the factorization of a difference of squares. If not, have students discover this identity by finding the factors of $5^2 - 2^2$, $9^2 - 4^2$, and $14^2 - 5^2$ and then relating them to the numbers being squared. Once the group is convinced that $m^2 - n^2 = (m + n)(m - n)$ ask them how we could prove this identity. With a picture, of course! Cutting and pasting is again required, as demonstrated below. Students may need some guidance to come up with the correct approach; start out by asking them how we could represent subtraction geometrically.



There are a host of applications involving the difference of two squares. Here are several problems which fit in with the spirit of the math circle as it has developed thus far. First, challenge students to compute the value of

$$20^2 + 19^2 + \cdots + 11^2 - 10^2 - 9^2 - \cdots - 1^2.$$

There are two different ways to pair up the squares into differences; for instance,

$$\begin{aligned} (20^2 - 10^2) + (19^2 - 9^2) + \cdots + (11^2 - 1^2) &= (30)(10) + (28)(10) + \cdots + (12)(10) \\ &= 10[(30 + 12) + (28 + 14) + \cdots] \\ &= 10(42 \cdot 5) = 2100. \end{aligned}$$

It is also entertaining to see whether students can compute $697^2 - 303^2$ in their head. (The answer is 394,000.) Another nice application is to ask students to factor the number 3599 by hand. If they get stuck, observe that this number appears “rigged.” Ask why, and then follow up by figuring out how to take advantage of this fact. Eventually students should determine that

$$3599 = 3600 - 1 = 60^2 - 1^2 = 59 \cdot 61.$$

Then see if they can factor 2491 on their own as $47 \cdot 53$.

Finally, no discussion of perfect square is truly complete without at least mentioning Pythagorean triples. Ask students if they can find two squares in our list which sum to another perfect square. The most obvious is $9 + 16 = 25$, of course; other examples include $36 + 64 = 100$ and $64 + 225 = 289$. However, explain that this is a very rich topic which deserves an entire math circle to itself to do it justice. So for today, we will pass it by in favor of another fascinating exploration.

Leave around half an hour for the final investigation. Begin by asking students to find a square number which is exactly twice another square. When they question whether this is possible, tell them that there is actually a single example that occurs early on in the list of perfect squares. If they still miss it, observe that $0^2 = 2(0^2)$, but that they are correct in thinking that there are no other such occurrences. Then inquire whether anybody can prove this fact. Note that stating that $\sqrt{2}$ is irrational doesn’t count as a proof; rather, it would be circular reasoning since this fact is based on the proof we are seeking. If no student is comfortable presenting a proof then develop one together by first considering how many factors of 2 can appear in a perfect square. (Technically we invoking unique factorization here, but it is probably wise not to raise this subtlety now.) By examining a few numbers in the list, students quickly realize that there are always an even number of 2’s in a positive perfect square, which makes sense given that a square is obtained by multiplying a number by itself. It is now clear that the equality $a^2 = 2b^2$ is impossible, since the left-hand side has an even number of factors of 2, while the right-hand side has an odd number.

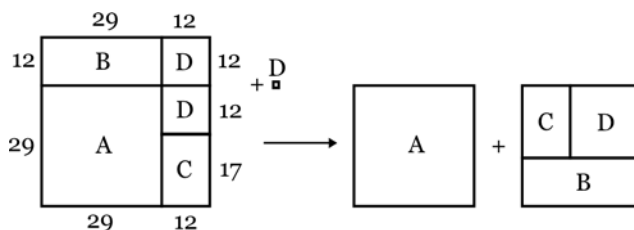
At this point the fun begins. In light of the above result, it makes sense to ask how close a perfect square can come to being equal to twice another square. The group will discover that the answer is “very close.” List their examples in a table such as the one shown below.

row	equality	squares	
n	$a_n^2 = 2b_n^2 \pm 1$	a_n	b_n
1	$1 = 2(1) - 1$	1	1
2	$9 = 2(4) + 1$	3	2
3	$49 = 2(25) - 1$	7	5
4	$289 = 2(144) + 1$	17	12

They may not stumble upon the first or fourth lines of the table immediately, but give them enough time and hints to complete the table. Having at least four rows makes it feasible to find patterns and make and test conjectures.

Then explain that we are looking for patterns which will enable us to predict the next two rows in the list. There are quite a few waiting to be found, the most obvious of which is the alternating values of $+1$ and -1 among the equalities. It is also true that $a_n + b_n = b_{n+1}$ and that $b_n + b_{n+1} = a_{n+1}$. Students will not phrase their observations in this manner, but it is a good exercise to practice writing a statement such as “adding the two numbers in any row gives the next b number,” using subscripts. These observations, or equivalent ones, are enough to generate the next two equalities $41^2 = 2(29^2) - 1$ and $99^2 = 2(70^2) + 1$. It is quite satisfying to check that these statements are valid. The recurrence relations $a_{n+1} = 2a_n + a_{n-1}$ and $b_{n+1} = 2b_n + b_{n-1}$ also hold, but wait to mention these until after the picture proof if students don’t notice them right away.

Emphasize that behind every clever pattern there is beautiful mathematics waiting to be discovered, and that understanding the pattern is what makes their pursuit worthwhile. In this case there are quite a few nice approaches to unraveling this particular mystery; we will employ a proof by picture, naturally. Ask what needs to be done geometrically to establish that $41^2 = 2(29^2) - 1$. Help students decide that it would suffice to dissect a 41×41 square, along with a single unit square, into two 29×29 squares. Let them tackle this problem for a while, displaying promising ideas and progress on the board. If they need a clue, suggest that we already know that two 12×12 squares plus a unit square are equivalent to a 17×17 square. Eventually the group should come up with a method such as the one pictured below.



Once they have found this dissection, walk them through the picture proof that $a_n^2 = 2b_n^2 - 1$ for any odd value of n . The same picture does the job, but now it becomes much more apparent where the pattern comes into play. If subscripts get in the way of understanding, appeal to the concrete case already established as you go. You could also mention that a similar picture will handle the case when n is even,

just place the single unit square on the other side of the arrow. Of course, this is an ideal place to broach the topic of induction. Point out that in order to guarantee that the dissection works for any particular row of the table, we needed to use the fact that it worked for the previous row. Since we have already confirmed the first several equalities, the rest are sure to follow.

Further Problems.

1. We have already seen that $213444 = 462^2$ is a perfect square. There is only one smaller perfect square that ends in three 4's. Can you find it?
2. Find the prime factorization of 8099 and 6319 without breaking a sweat.
3. How can the difference of two squares equal a prime? (A prime is a number such as 7, 23, or 41 which has no divisors other than 1 and itself.)
4. Find a positive integer which can be written as the sum of two perfect squares in two different ways. (There are four such numbers below 100, if we include 0 as a perfect square. See if you can find them all.)
5. Prove that a positive perfect square can never be equal to exactly three times another perfect square.
6. It is possible for three times a perfect square to differ from another perfect square by exactly one. For example, $3 \cdot 4^2 = 48$, which is just below $7^2 = 49$. Find at least three other examples of this sort of phenomenon. (By the way, it's permissible to use 0^2 .)
7. List your examples from the previous problem in increasing order. Can you detect a pattern which will allow you to generate more examples? Test your conjecture.
8. Find a picture proof that establishes the remarkable identity

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

(HINT: first find a way to represent a quantity such as 3^3 using squares.)

Hints and Answers.

1. Expand $(500 - 462)^2$ to see that this number and 462^2 have the same three final digits. Hence the answer is $38^2 = 1444$.
2. We have $8099 = 90^2 - 1^2 = 7 \cdot 13 \cdot 89$ and $6319 = 80^2 - 9^2 = 71 \cdot 89$.
3. If the quantity $(m+n)(m-n)$ is to be prime, one of the factors will have to be 1. This is only possible if $m = n + 1$ and $m + n$ is prime, such as $m = 12$ and $n = 11$.
4. They are $25 = 3^2 + 4^2 = 0^2 + 5^2$, $50 = 1^2 + 7^2 = 5^2 + 5^2$, $65 = 1^2 + 8^2 = 4^2 + 7^2$, and $85 = 2^2 + 9^2 = 6^2 + 7^2$.
5. Using the same reasoning as before, a perfect square a^2 will have an even number of factors of 3, while $3b^2$ will have an odd number of 3's, hence $a^2 = 3b^2$ is not possible.
6. There are also $1^2 = 3(0^2) + 1$, $2^2 = 3(1^2) + 1$, and $26^2 = 3(15^2) + 1$.
7. We have $+1$ in all of these equalities. Labeling the squares as a_n^2 and b_n^2 as before, we have $a_n + 2b_n = b_{n+1}$ and $2b_{n+1} - b_n = a_{n+1}$. There are also the recurrence

relations $a_{n+1} = 4a_n - a_{n-1}$ and $b_{n+1} = 4b_n - b_{n-1}$. Either way, the next row of the table should be $97^2 = 3(56^2) + 1$.

8. Think of 3^3 as three 3×3 squares, and similarly for the other cubes. To show that $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$ for example, draw nested 1×1 , 3×3 , and 6×6 squares all having a common bottom left vertex. Then fit a unit square, two 2×2 squares, and three 3×3 squares into the resulting L-shaped regions. You will need to cut and paste for the 2×2 squares.

Chapter 12

Making Change

Generating Functions (★ ★ ★)

Overview. The topic of generating functions typically elicits a combination of fascination and awe from students who have heard of them. This math circle sets out to give students first-hand experience with generating functions via a particularly nice application to a classic problem of Frobenius. The circle will be exhilarating for an advanced group that has at least ninety minutes to devote to the topic, but may be overly ambitious for other groups. In the latter case, cover the middle portion on the coin-exchange problem at a more leisurely and thorough manner (see the further problems) during one meeting, then present generating functions during the following meeting. Regardless, advise interested students to read the classic introduction to this subject, *Generatingfunctionology*, by Herbert Wilf. It is available for free download and can be found by typing the name of the book into any search engine. Finally, thanks to Matthias Beck at San Francisco State University for providing the engaging ideas, information, and problems on which much of this presentation is based.

Background. Imagine that we introduce a new coin system involving only 4-cent and 7-cent coins. One might object that certain amounts cannot be obtained using these coins; for example, 1, 2, or 5 cents. On the other hand, this deficiency makes our new coin system more interesting than the old one, because we can ask, “For which amounts are we able to make change?” A natural question, first tackled by Ferdinand Frobenius and James Sylvester in the nineteenth century, is “What is the largest amount that cannot be changed using coins with values a and b ?” This problem and its generalization to several coins is known as the *Frobenius coin-exchange problem*.

The simple-looking answer that will be found in the course of this math circle inspired a great deal of research into formulas in the general case, with limited success. While it is safe to say that the formula for two coins has been known for more than a century, no analogous formula exists for three or more coins. The formula we will deduce is due to Sylvester and was published as a math problem in the *Educational Times* more than a century ago. For more on the Frobenius problem, we refer the reader to the research monograph *The Diophantine Frobenius problem* by Jorge L. Ramírez-Alfonsín.

Problems.

1. Find a closed form expression for the following three infinite series.

- a. $1 + x + x^2 + x^3 + x^4 + \cdots$
- b. $1 + x^2 + x^4 + x^6 + x^8 + \cdots$
- c. $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

2. Find the coefficients of x^{14} and x^{17} in the expansion of

$$(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots).$$

3. Interpret the coefficient of x^n in the above product. In other words, complete the sentence, “The coefficient of x^n is the number of ways to _____.”

4. Create a product similar to the above expression in which the coefficient of x^n counts the number of ways to write n as the sum of a whole number and a positive odd integer.

5. Suppose that a certain country has a 3-cent coin and a 4-cent coin. Then it is not possible to obtain certain amounts of money using only these coins, such as five cents. Determine which amounts can be obtained with these coins.

6. Repeat the previous problem with a 3-cent and 5-cent coin. Then try it once more with a 3-cent and 7-cent coin.

7. Based on the results of the previous two problems, make a conjecture as to the largest amount that cannot be obtained using only 3-cent and b -cent coins for any value of b not divisible by 3. Also conjecture the number of amounts that cannot be obtained.

8. Explain why the situation changes when b is divisible by 3.

9. Prove that all amounts greater than or equal to $3b$ can be obtained using 3-cent and b -cent coins.

10. Let r_k be the number of ways to obtain exactly k cents using 2-cent and 3-cent coins. Make a table listing the values of r_k for $0 \leq k \leq 15$. What do you notice about this sequence?

11. Prove that $r_{k+6} = r_k + 1$ for all $k \geq 0$.

12. Demonstrate that the series $r_0 + r_1x + r_2x^2 + r_3x^3 + r_4x^4 + \cdots$ can be written

$$(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots),$$

by suitably interpreting the coefficient of x^k in the product.

13. Let $g(x)$ be the above product. Find a closed form expression for $g(x)$.

14. Show that $g(x) - x^6g(x)$ and $1 + x + x^2 + x^3 + x^4 + \cdots$ are identical series except for a single term. Which term, and why?

Presentation Notes. Tell your group that they are in for a treat today, because you are about to introduce them to generating functions: a rather clever, powerful means

of solving certain types of counting problems. One of the reasons that generating functions are so effective stems from the fact that they condense a large amount of information into a versatile, tidy package which can be strategically manipulated via algebra or calculus. To illustrate what you mean, ask students to describe the ten most elementary sequences they can come up with. Responses will vary, but may include the natural numbers, the squares, Fibonacci numbers, powers of two, and so on. These are all more sophisticated than desired, so next limit them to sequences involving only the numbers 0 or 1. Examples will hopefully include the sequences $\{1, 1, 1, 1, \dots\}$, $\{1, 0, 1, 0, \dots\}$, and $\{1, 1, 1, 0, 0, 0, \dots\}$; if not, be sure to include them in your list. (The sequence of all zeros is too trivial to be illuminating; when it is suggested ask if you can replace it with the latter sequence instead.)

Next explain that all the numbers in a sequence can be encoded into a single function of one variable (typically x) by inserting these numbers as the coefficients of increasing powers of x , beginning with a constant term. Demonstrate how to encode the sequence of all 1's as

$$1 + 1 \cdot x + 1 \cdot x^2 + 1 \cdot x^3 + \dots,$$

then have the group create series for some of their other examples. The two remaining sequences above become $1 + x^2 + x^4 + x^6 + \dots$ and $1 + x + x^2$. Remark that the latter expression has a special name—see if anyone can identify it as a polynomial. After creating a variety of series, the next step is to unveil the “tidy package” that was advertised earlier. Ask if anyone knows a more compact way to write any of the series just created, and have a student present an argument akin to

$$\begin{array}{rcl} g(x) & = & 1 + x + x^2 + x^3 + x^4 + \dots \\ - \quad xg(x) & = & x + x^2 + x^3 + x^4 + \dots \\ \hline (1-x)g(x) & = & 1 \end{array}$$

Therefore we conclude that our sequence of all 1's can be compactly encoded by the formula

$$g(x) = \frac{1}{1-x}.$$

Use a similar method to deduce the closed form expression for the generating function of $\{1, 0, 1, 0, \dots\}$, which is $1/(1-x^2)$. Then ask students how they could have obtained this formula in one quick step using the previous result. (Just replace x by x^2 everywhere.) See if they have caught on by having them find the generating function for $\{1, 0, 0, 1, 0, 0, 1, \dots\}$ in their heads. (Replacing x by x^3 does the trick, giving the answer $1/(1-x^3)$.) Finally, have students tackle the sequence $\{1, 2, 3, 4, \dots\}$ of counting numbers using the same $g(x) - xg(x)$ technique. They should conclude that its generating function is $g(x) = 1/(1-x)^2$. If appropriate for the audience, point out that there is also a shortcut for obtaining the series $1 + 2x + 3x^2 + \dots$ from the series $1 + x + x^2 + x^3 + \dots$; perhaps some student will suggest differentiation. The derivative of $1/(1-x)$ then gives the previous formula immediately.

The issue of convergence is likely to have come up by now—some properly taught calculus student may remark that these formulas are only valid for values of x in the range $|x| < 1$. Note that this is true, if one wishes to evaluate them at certain values of x . But explain firmly that there is a solid mathematical foundation for treating these expressions purely as algebraic objects, in which case all the subsequent manipulations can be made without worrying an iota about convergence. Avoid becoming further

embroiled in discussions of convergence if possible; instead invite interested students to meet with you afterwards to hear about formal power series and the field $\mathbb{Q}((x))$ of quotients of series with rational coefficients.

Now that the group has mastered a few basic generating functions, it is time to put them to work. (The students and the functions.) Have them each pick their favorite positive integer from 10 to 20. Then have them compute the coefficient of x to that power in the product

$$(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots).$$

To get them started, announce that you have chosen 9 (contrary to your own instructions), and observe that there are five different ways to obtain an x^9 term, namely $x \cdot x^8$, $x^3 \cdot x^6$, $\dots$, $x^9 \cdot 1$. Hence there will be a $5x^9$ term in the product. When they are finished students can compare figures, but explain that the value of the exercise is in the process rather than the answer. Ask students to interpret the coefficient using complete sentences. In other words, fill in the blank:

The coefficient of x^n is the number of ways to _____.

Eventually everyone should agree that the statement should be complete, “ways to write n as the sum of a whole number and an even whole number.” This is one of the pivotal moments in the circle, so give them as much time as needed to make this connection without belaboring the point.

To drive home this connection, you can follow up by directing students to write down an expression in which the coefficient of x^n is the number of ways to write n as the sum of a whole number and an odd whole number. (Just replace the second factor above by $x + x^3 + x^5 + \cdots$.) We are now in a position to prove our first theorem using generating functions. Inquire as to how one should combine the two expressions just derived to find the number of ways to write n in either one manner or the other. (Add them.) Now things become exciting, at least for the algebra fans in the group. Replace the series by their closed form expressions and perform the algebra to discover that

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} + \frac{1}{1-x} \cdot \frac{x}{1-x^2} = \frac{1+x}{(1-x)(1-x^2)} = \frac{1}{(1-x)^2}.$$

The latter expression is the generating function for the number of ways to write n as the sum of a whole number and either an even or odd whole number. But we already know the coefficient of x^n in that expression! Hence we have proved that there are precisely $(n+1)$ ways to write n in the manner described. This makes perfect sense, upon further reflection; we are really just asking for the number of ways to write n as the sum of two whole numbers.

It is time to shift gears slightly by presenting the Frobenius coin-exchange problem. After describing this problem (see the overview above) let students explore a bit on their own. On the board write down the ten pairs of coin denominations listed below and assign one per student so that each pair is taken by at least one (and preferably several) of the students.

$$(a, b) = (2, 3) \ (2, 4) \ (2, 5) \ (2, 7) \ (3, 4) \ (3, 5) \ (3, 7) \ (4, 5) \ (4, 6) \ (6, 9)$$

Their job is to investigate the possibility of making change using coins with their particular values of a and b . (Deliberately keep your directions a little vague.) Give the group a few minutes to work, directing students to try out another pair of coins if they are satisfied with their work. Rather than sharing the details of their investigation,

instead invite students to formulate questions that could be asked about this process. More precisely, think of some aspect of this investigation which each person could quantify or measure using their own pair of coins. As much as possible coax the students to verbalize questions, since this is an important mathematical skill with which they have had little or no practice. Their list might include questions such as

- How many amounts cannot be changed using the given two coins?
- What is the largest (or smallest) amount that cannot be changed?
- What is the smallest amount that can be changed in two different ways?
- In how many ways can a particular amount be changed?

Try to ensure that the four items above appear in your list. To motivate the final question, note that it seems to be guaranteed that large amounts can be changed, but there is still something interesting that can be asked in this case. Along the way students will probably make other observations or conjectures, so keep a parallel list of statements like

- The problem is trivial if one of the coins has value 1.
- If the numbers a and b have a common factor d , then the analysis is essentially the same as for the pair of coins with values $\frac{a}{d}$ and $\frac{b}{d}$. Hence it suffices to study the cases in which a and b are relatively prime.
- After some point all amounts can be changed.

From here the conversation becomes very unscripted—enjoy it. However the discussion unfolds, there are two important points to make. First, highlight the observation regarding common factors of a and b , and state that from here on we will assume that a and b are relatively prime. Secondly, bring the focus around to the question of how many ways there are to change a particular amount. Suggest that this line of inquiry subsumes all the others, in the sense that understanding it will allow us to more or less completely understand the Frobenius coin-exchange problem.

Therefore let us define r_k as the number of ways in which the amount k can be changed using coins of given (relatively prime) denominations a and b . Study the case $a = 2$, $b = 3$ to begin, making a table such as the one below.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
r_k	1	0	1	1	1	1	2	1	2	2	2	2	3	2	3	3

Make sure that the group is comfortable with the fact that $r_0 = 1$; in other words, using no a -coins and no b -coins is the one legitimate way of obtaining an amount of zero. Ask students what they notice. It is fairly obvious that there is a repeating pattern; help students turn their observation into a precise statement like $r_{k+6} = r_k + 1$. In general, what do they suppose is true? The fundamental observation to be made is that $r_{k+ab} = r_k + 1$.

Depending on the time remaining, you could now have students establish the relationship $r_{k+ab} = r_k + 1$, or you could challenge them to prove it later. (You will need approximately twenty to thirty minutes to cover the subsequent paragraphs.) It is more transparent how to proceed when working with the variables a and b rather than the special case $a = 2$, $b = 3$, so have students try the general case right away. The basic idea is to argue that the list of possible ways to make change for the amount k will look like

$$k = ma + nb = (m - b)a + (n + a)b = (m - 2b)a + (n + 2a)b = \cdots,$$

beginning with the example involving the largest number m of a -coins. Then reason that the list of ways to make change for the amount $k + ab$ will be

$$k + ab = (m + b)a + nb = ma + (n + a)b = (m - b)a + (n + 2a)b = \cdots,$$

which permits exactly one more method of making change than before. Acknowledge that some details have been glossed over here, particularly the transition from $r_k = 0$ to $r_{k+ab} = 1$, and suggest that students fill them in on their own as an exercise.

Point out to the group that in listing the number of ways to change different amounts (the values r_k) we have created a sequence. What shall we do with it? Examine its generating function, of course! Returning to the special case $a = 2$, $b = 3$ for the time being, let us define

$$g(x) = \sum_{k=0}^{\infty} r_k x^k = 1 + x^2 + x^3 + x^4 + x^5 + 2x^6 + x^7 + 2x^8 + \cdots.$$

Ask students to think back to their interpretation of the coefficients in the product

$$(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)$$

as a clue to how to find a formula for $g(x)$. Remind them that r_k can be interpreted as the number of ways to write k as a sum of a non-negative multiple of a and a non-negative multiple of b . With luck, someone will suggest that

$$g(x) = (1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots) = \frac{1}{1 - x^2} \cdot \frac{1}{1 - x^3}.$$

Take some time to allow students to convince themselves of this fact. Then let them predict that the generating function for arbitrary denominations a and b is

$$g_{a,b}(x) = \frac{1}{1 - x^a} \cdot \frac{1}{1 - x^b}.$$

Now ask the group how we can take advantage of the “periodic” nature of the values of r_k . Again, remind them of how we handled the series $1 + 2x + 3x^2 + 4x^3 + \cdots$ by subtracting x times this series from itself. A sharp student may realize that we should try computing $g(x) - x^6 g(x)$. Otherwise, remark that in the series $1 + 2x + 3x^2 + 4x^3 + \cdots$ each coefficient is one greater than the coefficient immediately preceding it, while in our series for $g(x)$ each coefficient is one greater than the coefficient six terms earlier, so how shall we modify our subtraction step? Eventually the group should agree that

$$g(x) - x^6 g(x) = 1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots,$$

where all terms beyond x^6 are guaranteed to have a coefficient of 1 since the relationship $r_{k+6} = r_k + 1$ applies. Ask students why every power of x appears except the x^1 term. (Because this is the single amount that is not changeable using coins of value 2 and 3.) Then generalize to arbitrary coins, with the result that

$$g_{a,b}(x) - x^{ab} g_{a,b}(x) = \frac{1 - x^{ab}}{(1 - x^a)(1 - x^b)}$$

is a series consisting of exactly those powers of x (with coefficient 1) which can be changed using coins of denomination a and b .

Finally, ask students how we could obtain an expression that involves only those powers of x corresponding to amounts that cannot be changed using coins with values

a and b . Hopefully it will occur to someone to subtract the above series from the series $1 + x + x^2 + x^3 + \cdots$. Otherwise ask how the latter series could help accomplish the task. We have now arrived at our major result for the day: the expression

$$\frac{1}{1-x} - \frac{1-x^{ab}}{(1-x^a)(1-x^b)} = \frac{x - x^a - x^b + x^{a+b} + x^{ab} - x^{ab+1}}{(1-x)(1-x^a)(1-x^b)}$$

contains the term x^k for exactly those values of k which cannot be changed. Have students check this formula when $a = 2$, $b = 3$ by expanding the denominator in this case. They should discover that the entire fraction simplifies to x^1 , as predicted. Next inquire as to how many terms will appear in this expression. (One for each non-changeable amount; in particular, a finite number.) Stress that we have deduced that the above expression is actually a polynomial! In other words, the factors in the denominator divide evenly into the numerator, which is a noteworthy fact by itself.

The generating function and polynomial developed above can be used to answer just about any question that one might care to ask about amounts which can or cannot be changed using coins of denomination a and b . Ask the group what aspect of the generating function or polynomial will answer each of the questions they raised earlier. For example, the largest non-changeable amount is the degree of the polynomial, which can be calculated by subtracting the degree of the denominator from that of the numerator. (The answer is $ab - a - b$.) The smallest amount that can be changed in two different ways is $k = ab$. And the number of non-changeable amounts is simple to compute, in theory: simply substitute $x = 1$ into the polynomial. In practice this requires some effort, since $x = 1$ is a triple root of both the numerator and denominator of the rational function above. But with three applications of L'Hôpital's Rule and a fair amount of algebra one discovers that exactly $\frac{1}{2}(a-1)(b-1)$ amounts cannot be changed. These results can be summarized quite nicely by observing that every non-changeable amount appears in list

$$0, 1, 2, 3, \dots, ab - a - b - 1, ab - a - b,$$

these amounts account for exactly half the numbers in the list, and the largest non-changeable amount is the final number.

Further Problems. In the following problems we develop a more elementary approach to solving the Frobenius coin-exchange problem for two coins.

1. Find integers m and n such that $4m + 7n = 1$. Then find another pair of values for m and n such that $4m + 7n = 1$ again.
2. If k is a given positive integer, show that we can always find integers m and n such that $4m + 7n = k$. Then demonstrate that this is still possible if we further require that $0 \leq m \leq 6$.
3. Show that the following recipe works for determining whether or not a given amount k can be changed using 4-cent and 7-cent coins. Given k , find integers m and n such that $4m + 7n = k$ and $0 \leq m \leq 6$. Then k can be changed precisely if $n \geq 0$.
4. Use the idea outlined in the previous problem to determine the largest amount that cannot be obtained using 4-cent and 7-cent coins.
5. Let a and b be relatively prime positive integers. Generalize the reasoning developed in the preceding problems to analyze the case of two coins worth a cents

and b cents. You may use the fact that the Euclidean algorithm guarantees the existence of integers m and n such that $am + bn = 1$.

6. Suppose k is an integer between 0 and ab that is not a multiple of a or b . Prove that if the amount k can be changed then $ab - k$ cannot be changed, and conversely, if k cannot be changed then $ab - k$ can be changed.

7. Prove that there are exactly $\frac{1}{2}(a-1)(b-1)$ amounts that cannot be changed.

8. Prove that if the positive integers a , b , and c have no common factor then there is some largest amount that cannot be changed using coins worth a , b , and c cents. In other words, show that after some point all amounts can be changed. (We are assuming that a , b , and c are not all divisible by some integer $d \geq 2$. However, any two of them might have a common factor, as is the case for $a = 6$, $b = 10$, and $c = 15$.)

Hints and Answers.

1. The pairs $(m, n) = (2, -1)$ or $(9, -5)$ or $(-5, 3)$ all work.

2. Take $m = 2k$ and $n = -k$. Then repeatedly subtract 7 from m and add 4 to n , which does not affect the value of $4m + 7n$, until m is in the desired range.

3. Clearly if $n \geq 0$ then we have found a way to obtain k cents. Conversely, if $n < 0$ then argue that the “next” solution is found by subtracting 7 from m and adding 4 to n , which results in $m < 0$. Hence there is no solution to $4m + 7n = k$ satisfying $m \geq 0$ and $n \geq 0$.

4. Write $4m + 7n = k$ with $0 \leq m \leq 6$ as before. If k cents cannot be changed then $n < 0$. But the largest possible values for m and n in this case are $m = 6$ and $n = -1$, yielding $k = 4(6) + 7(-1) = 17$ as the maximal amount that cannot be obtained.

5. The same reasoning reveals that $k = a(b-1) + b(-1) = ab - a - b$ is the largest non-changeable amount.

6. Write $k = am + bn$ with $0 \leq m \leq b-1$. If k can be changed then $n \geq 0$. Next note that in fact $m > 0$ and $n > 0$, since we are assuming that k is not a multiple of a or b . Then observe that

$$ab - k = ab - (am + bn) = a(b - m) + b(-n).$$

Now $b - m$ is in the range $0 \leq b - m \leq b - 1$ and $-n < 0$, hence $ab - k$ cannot be changed. The converse is proven similarly.

7. There are exactly $ab - a - b + 1 = (a-1)(b-a)$ values of k from 0 to ab that are not multiples of a or b . The previous problem implies that exactly half of these may be changed. (Clearly any multiple of a or b may be changed.)

8. One simple argument involves showing that a is relatively prime to $b + c$. By using one b -cent coin and one c -cent coin we can effectively create a $(b + c)$ -cent coin. Now use the fact that all amounts above $a(b + c) - a - (b + c)$ can be changed.

Chapter 13

Two Squares Are Better Than One

Suave numbers ($\star\star$) OR
Gaussian integers ($\star\star\star$)

Overview. The problem of which integers may be written as the sum of two squares is the classic entry point to the study of algebraic number theory. This question lends itself well to a math circle investigation since there are many readily apparent patterns to be noticed among such numbers. Furthermore, their proofs can often be found or at least understood by secondary school students. But this topic can also lead to deeper waters: the central observation that all primes one greater than a multiple of four can be expressed as the sum of two squares is a mysterious fact which cannot quickly be explained. One of the most elegant demonstrations relies on the ring of Gaussian integers; that is, numbers of the form $a + bi$ where a and b are ordinary integers. Gauss studied these sorts of numbers while attempting to formulate and prove higher order reciprocity laws, following his success with quadratic reciprocity.

Problems. We define a *suave* number as any integer which can be written as the sum of two perfect squares. Thus n is suave if and only if we can write $n = a^2 + b^2$ for integers a and b . Hence the first several suave numbers are 0, 1, 2, 4, 5, 8, 9, and 10.

1. Confirm that the numbers just listed are in fact all suave. Then explain why 3, 6, and 7 are not suave. Extend the above list to include all suave numbers from 0 to 50 (or 100, if desired). Why are there no negative suave numbers?
2. By examining this list, confirm that the product of two suave numbers is also suave. (No need to prove this statement yet.) Is the product of two non-suave numbers always non-suave?
3. Prove that if n is suave, then so is $2n$. (HINT: suppose that we can write $n = a^2 + b^2$. Then we must have $2n = (\quad)^2 + (\quad)^2$ in order for $2n$ to be suave. Write expressions inside the parentheses to ensure that this expression simplifies to $2a^2 + 2b^2 = 2n$.)

4. Using the same approach as in the previous problem, show that if n is suave, then so is $5n$.
5. Generalize the algebraic pattern begun here to prove that if m and n are both suave, then so is mn .
6. Prove that if n is a positive suave number, then $3n$ cannot be a suave number. (IDEA: suppose to the contrary that there do exist positive suave numbers n for which $3n$ is also suave, and let N be the smallest. Thus we can write $N = a^2 + b^2$ and $3N = c^2 + d^2$. Now explain how to find a smaller example by proving that both c and d must be divisible by 3.)
7. In the same manner, demonstrate that if n is a positive suave number, then $7n$ cannot be suave.
8. Show that any suave number must contain an even number of factors of 3.
9. By examining your list of suave numbers, make a conjecture regarding which prime numbers are suave.
10. Develop a simple algorithm based on the prime factorization of a given number n to determine whether n is suave. Use your algorithm to check whether 9999 is suave without adding together any square numbers.

Presentation Notes. Begin by declaring that, as the title suggests, two squares are better than one. To be precise, sums of two squares give rise to an amazingly rich mathematical theory well worth investigating. So ask students, “Which integers can be written as the sum of two perfect squares? In other words, for which n do we have $n = a^2 + b^2$ for integers a and b ?” Entertain suggestions for a few moments. Initiate a debate (if it does not begin spontaneously) as to whether perfect squares such as 9 should be included in the list. Zero is the square of an integer, so students should agree that $9 = 0^2 + 3^2$ does qualify—so does 0, for that matter. Then let participants coin an adjective to describe integers that can be written as the sum of two squares. In this presentation we refer to such numbers as “suave,” but allow students to take a degree of ownership of the topic by deciding as a group on their own term.

In order to begin exploring properties of suave numbers, it will help to have a fair-sized list of them. Have the group determine all suave numbers from 0 to 100 and write their list on a reserved portion of the board. Organize them if necessary, otherwise indicate that they should figure out how to accomplish the task efficiently themselves. Make a mental note of mistakes or omissions but don’t point them out immediately; rather, let students catch them during the course of the presentation. Once the group is satisfied with their list and everyone has returned to their seats, instruct students to examine the list and make as many observations or conjectures related to suave numbers as they can find. These might include statements such as

- a. Negative integers cannot be suave.
- b. If n is suave, then so are $2n$ and $5n$.
- c. If n is a positive suave number, then $3n$ and $7n$ are not suave.
- d. If a suave number is divisible by 3, then it must be divisible by 9.
- e. More precisely, a positive suave number must have an even number of factors of 3 and 7.
- f. The product of two suave numbers is also suave.

These conjectures would make for a pleasant discussion by themselves. If the group has only an elementary understanding of number theory, consider spending the bulk of your time proving the above assertions, then summarizing the more advanced material that follows as a matter of general interest. Alternatively, if participants are familiar with congruences, for example, then relegate these basic considerations to a supplementary problem set in favor of using the available time to discover the beautiful connection with complex numbers outlined further on.

Depending on your style, either tackle the observations made in a structured, sequential manner or let students work on whichever statements seem the most interesting to them while you (and assistants, if possible) circulate to hear and comment on their progress. Both approaches can be effective; for sake of exposition, we will adopt the former. To begin, the first statement is clear. Help students formulate the second statement algebraically before they attempt to prove it; something along the lines of, “Suppose that n is suave, with $n = a^2 + b^2$. Then $2n$ is also suave, because $2n = (\quad)^2 + (\quad)^2$.” The task is to fill the parentheses with algebraic expressions so that the result simplifies to $2(a^2 + b^2)$. The group could also employ an inductive reasoning approach by listing several examples and finding the pattern; either way they should eventually surmise and then verify that $(a + b)^2 + (a - b)^2$ does the job. See if they can find the algebraic explanation to establish the same result for $5n$. It should not be difficult to discover that $(2a + b)^2 + (a - 2b)^2 = 5(a^2 + b^2)$, which proves that if $n = a^2 + b^2$ is suave then $5n$ is also.

On the other hand, direct observation suggests that if n is a positive suave number then $3n$ is not suave. (Note that $n = 0$ would be an exception if we did not restrict our attention to positive numbers.) Ask students how they might go about proving that something cannot happen. (Employ proof by contradiction.) So assume that n and $3n$ are both suave, say $n = a^2 + b^2$ and $3n = c^2 + d^2$. Let students experiment with the second statement—how can two perfect squares sum to a multiple of 3? They should conclude that this occurs only if both c and d are themselves multiples of 3. To prove this observation, have students verify that when c is not a multiple of 3, then c^2 is 1 more than a multiple of 3. One could use the language of congruences, i.e. $c \equiv 2 \pmod{3} \implies c^2 \equiv 4 \equiv 1 \pmod{3}$. For less advanced students, simply observe that if $c = 3k + 2$ then $c^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Regardless, agree that c^2 and d^2 are either a multiple of 3 or 1 greater than a multiple of 3. Then try all possible combinations to conclude the proof.

In summary, if $3n = c^2 + d^2$ then both c and d must themselves be multiples of 3, so we can write $c = 3c'$ and $d = 3d'$. This leads to $3n = 9(c')^2 + 9(d')^2$, or $n = 3(c')^2 + 3(d')^2$, meaning that n is also a multiple of 3, say $n = 3n'$. Let the group ponder the implications of this for a while, directing their attention to the as yet unused statement $n = a^2 + b^2$ if needed. They should realize that the same reasoning as before implies that both a and b must be multiples of 3, say $a = 3a'$ and $b = 3b'$. Making all the substitutions reduces to $n' = (a')^2 + (b')^2$ and $3n' = (c')^2 + (d')^2$. In other words, it appears that we are right back to where we started! Press students to identify the difference between this pair of statements and the original (unprimed) statements. It is crucial that they appreciate that these equations have the same form, but involve smaller (by a factor of 3) numbers. In other words, whenever we have a suave number n with the property that $3n$ is also suave, then we may deduce that $n = 3n'$ is a multiple of 3, and that the smaller number n' is also a suave number with the same property.

Ask everyone what should be done now. (Repeat the process, of course, to conclude that n' must itself be a multiple of 3, say $n' = 3n''$, and that n'' is also suave.) Now inquire why this situation is problematic. Give students enough time to realize that it is impossible to continue this process indefinitely, since no positive integer has infinitely many factors of 3. (Note that we can divide 0 arbitrarily many times by 3, though. Hence the exception alluded to earlier.) But our reasoning guarantees that we are able to keep dividing a factor of 3 out of n . Hence we must abandon our original assumption that both n and $3n$ are suave, since it leads us to the absurd conclusion that n has infinitely many factors of 3.

The concept of infinite descent (also known as *reductio ad absurdum*) is tricky for many kids to grasp the first time they encounter it. However, it is usually unwise to attempt to alleviate confusion by reviewing the entire argument; invariably those students will be no closer to understanding the concept than before, at the cost of losing momentum for the rest of the group. It might be worthwhile, though, to briefly sketch an alternate, equivalent formulation. Of all positive integers n with the property that both n and $3n$ are suave, let N be the smallest such number. (Invoking the well-ordering principle.) By our previous argument, we can then produce a smaller integer $N' = N/3$ with the same property, contradicting the minimality of N . In any case, after presenting the various arguments clearly and allowing sufficient time for students to digest the ideas, it is best to move on.

The same ideas will establish that n and $7n$ cannot both be positive suave numbers. Students could work on this problem right away, or try it at a later point. Perhaps the speaker could advertise a small prize for the nicest written solution that is received during the following week. (Students would be expected to submit only original work, without using outside references, of course.) The previous arguments also serve to establish the next two conjectures from the above list in a relatively clear manner.

There are two nice ways to wrap up this sequence of ideas. First, propose the natural question, “If n is a suave number, then which multiples of n are also suave?” So far we have seen that $2n$ and $5n$ are, while $3n$ and $7n$ are not. Allow the group to notice that 2 and 5 are themselves suave, while 3 and 7 are not. Hence we reach a two part conjecture; namely, if m and n are both suave, then so is mn . On the other hand, if n is suave but m is not, then mn is not suave either. The former follows from a general algebraic identity that students might be able to discover based on their earlier work: if $n = a^2 + b^2$ and $m = c^2 + d^2$, then

$$mn = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2.$$

The latter part of the conjecture requires a bit more machinery to prove in general and could be left as a challenge to motivated participants. Note that if neither m nor n is suave then the status of mn is in general unknown; for example $3 \cdot 7 = 21$ is still not suave, but $3 \cdot 3$ is suave.

The second satisfying conclusion that may be reached is the development of a general test for suaveness based on the prime factorization of n . (That such a test exists in the first place is remarkable, since suaveness is determined by sums, yet can be checked via factors!) Based on all that has been done students should hopefully be able to verbalize the following algorithm. First write n as a product of primes, then identify any non-suave prime factors, such as 3 or 7. If these primes occur an even number of times in the factorization, then n is suave; otherwise, n is not suave. If time permits, see if the group can discover a quick method for determining which primes are suave. The key is to compare the primes to the nearest multiple of four;

if $p = 2$ or if p is 1 more than a multiple of 4, then p is suave. But if p is 1 less than a multiple of 4, then p is not suave. The latter is not hard to show, but the former is an intriguing fact which cannot be quickly explained. For now, students should be content to marvel at this phenomenon and accept it without proof. They should also now be able to predict whether 9999 is suave. Since $9999 = 3^2 \cdot 11 \cdot 101$, the answer is no, without even adding together any square numbers!

If one wishes to explore the realm of Gaussian integers, then much of the preceding discussion should be omitted. After creating the list of suave numbers and soliciting observations, simply confirm conjectures and suggest that students prove them at a later point. However, it is worth spending a few moments looking for an algorithm to test whether a given number n is suave based on its prime factorization. Students should gradually realize that the matter hinges upon knowing which primes are suave, as explained in the preceding paragraph. Then ask them to find a quick means of checking whether a given prime p is suave. They should discover that the primes $p = 2$ and $p \equiv 1 \pmod{4}$ are suave, while primes $p \equiv 3 \pmod{4}$ are not. One can easily verify that $a^2 + b^2 \equiv 0, 1, \text{ or } 2 \pmod{4}$, confirming the latter statement. But the fact that all primes $p \equiv 1 \pmod{4}$ can be written as the sum of two squares is a much deeper result that is well worth understanding. For now, choose a few primes one more than a multiple of four, such as 53, 97, or 181, and confirm that they can be written as a sum of two squares.

One of the most elegant approaches to analyzing sums of two squares involves the ring of Gaussian integers. These consist of all complex numbers of the form $a + bi$, where a and b are ordinary integers and $i^2 = -1$. Have the group generate a variety of such numbers; include some purely real and imaginary numbers such as -1 and $3i$, for example. Once seven or eight numbers have been written down, ask students what sorts of operations can be performed with these numbers. Addition, subtraction, and multiplication are likely suggestions—in each case, choose two numbers already on the board and have everyone try out the given operation. Students should need no formal explanation of complex arithmetic, as long as they keep in mind that $i^2 = -1$. Have students help one another until all participants agree on each answer.

Division, of course, is another matter. Have a volunteer with some experience divide two complex numbers at the board. Point out that unlike the previous three operations, the quotient of two Gaussian integers is not necessarily another Gaussian integer. For this reason division will not play a major role in the proceedings. However, the concept of the conjugate, which was used to clear the i from the denominator in the division computation, will come in handy. State that if $\alpha = a + bi$ then its conjugate is given by $\bar{\alpha} = a - bi$. (Note the convention of using Greek lowercase letters for Gaussian integers.) Mention in passing that $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$ and that $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$. Have students compute $\alpha + \bar{\alpha} = 2a$ and $\alpha\bar{\alpha} = a^2 + b^2$, noting that each answer is an ordinary integer. The connection with sums of squares now becomes evident.

At this point introduce the norm of a Gaussian integer as $N(\alpha) = \alpha\bar{\alpha} = a^2 + b^2$, explaining that the norm measures the size of α , in some sense. (In fact, it is the square of the distance from α to the origin in the complex plane, although geometric considerations will not make much of an appearance with this topic.) Return to the sample multiplication problem from above and have students compute the norm of each of the factors as well as of the product. An observant participant will hopefully notice that $N(\alpha)N(\beta) = N(\alpha\beta)$. Have the group prove this important fact. Either

definition of norm will do, although conjugates are much neater:

$$N(\alpha\beta) = \alpha\beta \cdot \overline{\alpha\beta} = \alpha\beta\bar{\alpha}\bar{\beta} = \alpha\bar{\alpha}\beta\bar{\beta} = N(\alpha)N(\beta).$$

To help solidify the connection between norms and sums of squares, have students find all Gaussian integers α for which $N(\alpha) = 5$. (There are eight such numbers: $\pm 1 \pm 2i$ and $\pm 2 \pm i$.) Repeat this exercise with $N(\alpha) = 7$ (no such numbers, since 7 is not suave), and with $N(\alpha) = 25$ (this time there are twelve answers: $\pm 3 \pm 4i$, $\pm 4 \pm 3i$, ± 5 , and $\pm 5i$). Make sure that students appreciate that $N(5) = 5 \cdot \bar{5} = 5 \cdot 5 = 25$; in other words, $N(5) \neq 5$. Conclude with the important example $N(\alpha) = 1$, for which there are exactly four answers: 1, -1 , i , and $-i$. These numbers are known as *units* and play the same role in the Gaussian integers that 1 and -1 play in the ordinary integers, in that they are the only integers that divide all the others.

Recall that the building blocks of the ordinary integers are the primes—every non-zero integer can be written in an essentially unique way as a product of primes. Point out that $-10 = (2)(-5) = (-2)(5)$ does not really count as two different factorizations, since the same primes were used; the pesky -1 can float around if it likes. Now let students try their hand at factoring some Gaussian integers, such as $1 + 3i$ and 13. Find several “different” factorizations if possible, then confirm that the same factors actually appear, differing only by one of the four units. Answers might include

$$1 + 3i = (1 + i)(2 + i) = (-1 + i)(1 - 2i), \quad 13 = (2 + 3i)(2 - 3i) = (3 + 2i)(3 - 2i).$$

The norm is a rather convenient tool for searching for factors. See if students can figure out how to use it to their advantage when factoring $8 + i$. (We know that $N(8 + i) = 65$, so if $8 + i = \alpha\beta$ then we can deduce that $N(\alpha)N(\beta) = 65$. Hence we try $N(\alpha) = 5$ and $N(\beta) = 13$, since letting either norm equal 1 gives a trivial unit factor. A little experimentation gives $8 + i = (2 - i)(3 + 2i)$ as one possible answer.)

It is very instructive to allow the group to formulate their own definition of a Gaussian prime, usually denoted by the letter π . Help students refine their suggestions until they reach a satisfactory statement, such as “We say that $\pi \neq 0$ is a Gaussian prime if it is impossible to write $\pi = \alpha\beta$ for Gaussian integers α and β , neither of which are units.” The final condition could also be written, “with $N(\alpha) > 1$ and $N(\beta) > 1$.” It is natural at this point to look for a few concrete examples of Gaussian primes. Let students propose a couple of potential primes, then focus attention on a number such as $3 + 2i$. Ask what tool is likely to be helpful in ascertaining its primality, and let the group work out an explanation. (The norm helps, as usual. If $3 + 2i = \alpha\beta$, then $N(\alpha)N(\beta) = 13$, hence either $N(\alpha) = 1$ or $N(\beta) = 1$, so any “factorization” of $3 + 2i$ must involve a unit, thereby proving that $3 + 2i$ is indeed a prime.)

Now direct their attention to the ordinary primes 2, 3, 5, 7, 11, 13, ...; which of these are Gaussian primes? The pump has already been primed, so to speak; it shouldn't take long for participants to conjecture that when $p \equiv 3 \pmod{4}$ then p is a Gaussian prime, but when $p \equiv 1 \pmod{4}$ then p factors. Note that $p = 2$ is a special case; it actually does factor as $2 = (1 - i)(1 + i)$. Next show that 7 is prime using the norm function. (Suppose that $7 = \alpha\beta$, so that $N(\alpha)N(\beta) = 49$. We have already seen that there is no α for which $N(\alpha) = 7$, so we must have $N(\alpha) = 1$ and $N(\beta) = 49$ or vice-versa. In other words, there is no non-trivial way to factor 7.) An entirely analogous argument will show that p is a Gaussian prime whenever $p \equiv 3 \pmod{4}$.

So let p be an ordinary prime with $p \equiv 1 \pmod{4}$. The centerpiece of the entire argument that p can be written as the sum of two squares consists of showing that p is not a Gaussian prime. The chain of reasoning presented below is rather slick and

involves a couple of handy results about ordinary primes which may not be familiar to everyone. For this reason it is advisable to briefly walk through the argument, asking students for input and small calculations along the way. It may also help to review the steps with a particular prime, such as $p = 13$. Regardless, here is the general proof. To begin, let n be the ordinary positive integer $(\frac{p-1}{2})!$, i.e. $n = (1)(2)(3) \cdots (\frac{p-1}{2})$. We claim that $n^2 + 1$ is a multiple of p . To see why, first note that

$$\begin{aligned} n &= (-1)^{\frac{1}{2}(p-1)}(-1)(-2)(-3) \cdots \left(-\frac{p-1}{2}\right) \\ &\equiv (p-1)(p-2)(p-3) \cdots \left(\frac{p+1}{2}\right) \pmod{p}, \end{aligned}$$

using the fact that $\frac{1}{2}(p-1)$ is even. Therefore

$$\begin{aligned} n^2 + 1 &\equiv \left[(1)(2)(3) \cdots \left(\frac{p-1}{2}\right)\right] \left[(p-1)(p-2)(p-3) \cdots \left(\frac{p+1}{2}\right)\right] + 1 \\ &\equiv (p-1)! + 1 \\ &\equiv 0 \pmod{p}, \end{aligned}$$

as desired. In the last step we employed Wilson's theorem, which states that $(p-1)! \equiv -1 \pmod{p}$ for any prime p .

One of the defining characteristics of ordinary primes is that if a prime p divides the product ab of two integers then it must divide either a or b . This need not hold for composites; for instance, 6 divides $3 \cdot 4$, but clearly 6 does not divide either 3 or 4. The same property holds for Gaussian primes as well. But we have just seen that p divides $n^2 + 1 = (n+i)(n-i)$, and it is clear that neither factor is a multiple of p . (The lone i causes the trouble. For example, the multiples of 5 look like $15 - 10i$ or $5 + 20i$, and so on.) Therefore p cannot be a Gaussian prime! Pause for a moment to allow the full significance of this statement sink in, because the end of the argument sneaks up rather quickly now. It follows that we can write $p = \alpha\beta$ for some Gaussian integers α and β which are not units. Taking norms, we find that $p^2 = N(\alpha)N(\beta)$. Since $N(\alpha) \neq 1$ and $N(\beta) \neq 1$, it follows that $N(\alpha) = p$ and $N(\beta) = p$. Let us write α as $a + bi$. Then $N(\alpha) = p$ boils down to $p = a^2 + b^2$. And we're done!

In the interest of mathematical honesty, students should know that this presentation is not, strictly speaking, a complete proof of the fact that primes $p \equiv 1 \pmod{4}$ can always be written as the sum of two squares. At several key junctures we used properties of the Gaussian integers which were only stated without proof. For instance, we tacitly assumed that Gaussian integers have unique factorization, which does not necessarily hold for all rings of integers. This would follow once we showed that these numbers possessed a Euclidean algorithm. We also relied heavily on the fact that if $\pi|\alpha\beta$ then either $\pi|\alpha$ or $\pi|\beta$. This property also follows from the Euclidean algorithm.

The conclusion of this beautiful proof is, more than likely, the most logical place to end the math circle. However, there is still plenty of exciting mathematics remaining along these lines for interested students. For instance, one can investigate the number of ways that a given integer n may be written as the sum of two squares. There are nice patterns to be found in this direction that may be proved by considering the factorization of n into Gaussian primes. As a hint, first consider how many ways there are to write 5^k as the sum of two squares, then try products of distinct suave primes. Ambitious students might also wish to fill in the missing steps above by proving that the Gaussian integers possess a Euclidean algorithm.

Further Problems. The set of *Gaussian integers* consists of all complex numbers of the form $a + bi$, where a and b are ordinary integers and $i^2 = -1$. Thus $3 - 2i$, $7i$, and 4 are all Gaussian integers. The key to determining which ordinary integers can be written as the sum of two squares lies in understanding the Gaussian primes. We say that $\pi \neq 0$ is a Gaussian prime if it is impossible to write $\pi = \alpha\beta$ for Gaussian integers α and β , neither of which are units.

1. Let $\alpha = 10 - 11i$ and $\beta = 3 - 2i$. Compute $\alpha + \beta$, $\alpha - \beta$, $\alpha \cdot \beta$, and α/β .
2. If $\alpha = a + bi$ is a Gaussian integer, then its conjugate is $\bar{\alpha} = a - bi$. Confirm that both $\alpha + \bar{\alpha}$ and $\alpha\bar{\alpha}$ are ordinary integers.
3. Demonstrate that $\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$ and that $\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$.
4. We define the *norm* of a Gaussian integer $\alpha = a + bi$ as $N(\alpha) = a^2 + b^2$. (The norm measures how “large” α is, in some sense.) Determine $N(2 + 5i)$ and $N(1 - i)$. Now multiply $(2 + 5i)(1 - i)$, and compute the norm of the result. What do you notice?
5. Use properties of conjugates to prove that in general $N(\alpha\beta) = N(\alpha)N(\beta)$.
6. Find all Gaussian integers α for which $N(\alpha) = 1$. (There are four of them, called the units—every Gaussian integer is divisible by these four numbers.)
7. Now find all α such that $N(\alpha) = 5$. Then try $N(\alpha) = 7$ and $N(\alpha) = 25$. (There are twelve answers for the latter question. Don’t forget that $N(-5) = 25$, for example.)
8. Find a way to write $1 + 3i$ as a product of two non-trivial factors. In other words, don’t use the numbers 1 , -1 , i , or $-i$, which divide all Gaussian integers.
9. We know that 29 can be written as the sum of two squares. How does this lead to a way to factor 29 over the Gaussian integers?
10. Next find a way to write $2 + 9i$ as a product of two non-trivial factors. (HINT: use the norm.)
11. Show that $4 + i$ and 11 are prime Gaussian integers. (HINT: use the force. I mean the norm.)
12. Now suppose that $p \equiv 1 \pmod{4}$. (In other words, p is one greater than a multiple of four.) We know that $1 \cdot 2 \cdot 3 \cdots (p-2)(p-1) \equiv -1 \pmod{p}$ by Wilson’s theorem. Use this fact to argue that $[1 \cdot 2 \cdot 3 \cdots (\frac{p-1}{2})]^2 + 1$ is a multiple of p .
13. It is a fact that if a Gaussian prime divides $\alpha\beta$, then it must divide either α or β . The previous question implies that there is some number n such that $n^2 + 1$ is divisible by p . In other words, p divides $(n + i)(n - i)$. Explain why neither factor is divisible by p . Conclude that p is not a Gaussian prime.
14. We just saw that if $p \equiv 1 \pmod{4}$, then p is not prime in the Gaussian integers. Thus we can write $p = \alpha\beta$ for non-trivial numbers α and β . Explain why this means that $N(\alpha) = p$, and then deduce that $p = a^2 + b^2$ for some ordinary integers a and b . Thus p is suave!

Chapter 14

Reflection in a Circle

Inversive geometry (★ ★ ★)

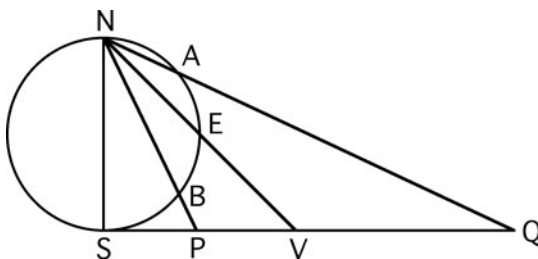
Overview. The purpose of this math circle is to provide a gentle introduction to the delightful subject of inversive geometry. The majority of sources credit the German mathematician L. J. Magnus with the “invention” of inversion in 1831, although it appears that a fair number of people deserve a portion of the credit, among them Steiner and Plücker. A reasonably complete description of this transformation and its properties first appears in 1836 in a paper by the Italian mathematician Giusto Bellavitis.¹

Although it is fair to say that inversion is a sophisticated geometric tool, the underlying ideas can certainly be grasped, appreciated, and applied by any student with a rudimentary background in plane geometry. A working knowledge of similar triangles and an understanding of the interplay between angles and circles is important, but no specialized knowledge beyond these basics is required. Still, it may be advisable to hold a math circle that focuses on elementary problems in plane geometry to help review these two topics the week prior to a math circle on inversion. As mentioned later in the presentation notes, it is also helpful to have a large ball on hand for illustrating stereographic projection; a beach ball would work well, for instance.

Problems. The usual set of problems is not as practical for this math circle. A sampling of the questions that are covered appear below and in the further problems, although they may be somewhat disjointed outside of the context of the discussion outlined in the presentation notes.

1. We perform a “reflection in a circle” by drawing a horizontal equator on the surface of a sphere and then swapping any pair of points which lie directly above and below one another. What are the fixed points of this transformation? What happens to a circle when it is reflected?
2. In the diagram below, points N , E , and S are the northern, eastern, and southernmost points of the circle, and $\overline{SV}$ is tangent to the circle at S . Show that $SN = SV$.

¹The Origins of the Geometric Principle of Inversion, by Boyd C. Patterson, 1933, The History of Science Society.



3. In the diagram points A and B lie directly above and below one another. If rays NA and NB are extended to meet the tangent line at points Q and P , then prove that $\triangle NSQ \sim \triangle PSN$. Use this fact to argue that $(SP)(SQ) = (SV)^2$.

We now briefly state the standard definition of inversion. Consider a circle in the plane with center S and radius r . An inversion of the plane through this circle takes any point P (other than S itself), to a point P' on ray $\overrightarrow{SP}$ such that $(SP)(SP') = r^2$. Finally, S is mapped to N , the point at “ N -finity,” and vice-versa.

4. Which points in the plane are mapped to themselves under an inversion?
5. What is the image of a line passing through S ? What is the image of a circle with center S and radius $2r$?
6. Suppose that points A and B are inverted to points A' and B' . Prove that $\triangle SAB \sim \triangle SB'A'$.
7. We can now deduce how distances are affected by an inversion. Using the notation from the previous problem, prove that

$$A'B' = AB \cdot \frac{r^2}{(SA)(SB)}.$$

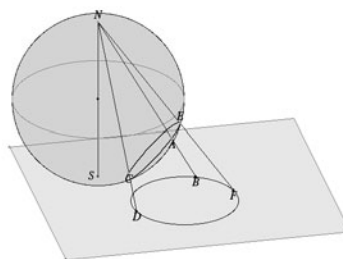
8. We have just seen that $\triangle SAB \sim \triangle SB'A'$. Consequently, which pairs of angles must be congruent? Now prove that a line not passing through S inverts to a circle through S . (HINT: let A , B , and C be three points along the line. Prove that A' , B' , C' , and S lie on the same circle by showing that $\angle A'SC'$ and $\angle A'B'C'$ are supplementary.)
9. Prove Ptolemy’s Theorem, which states that if A , B , C , and D are any four points in the plane, then $(AB)(CD) + (AD)(BC) \geq (AC)(BD)$, with equality if and only if the four points appear in alphabetical order around a single circle. (HINT: perform an inversion about a circle with center D and radius 1, then apply the triangle inequality to points A' , B' , and C' .)
10. Let S be a point outside a given circle, and let T be a point on the circle such that line ST is tangent to the circle. Prove that an inversion through the circle with center S and radius ST leaves the original circle fixed. (The points on the circle will move around, but the circle as a whole returns to precisely the same position.)

Presentation Notes. To begin the discussion on familiar territory, draw a straight line on the board and ask students what it means to perform a reflection in (or over, or across) this line. Responses should include the idea that each point of the plane is mapped to another point—or possibly to the same point, if it happens to lie on the line of reflection. The correspondence can be visualized by spinning the plane 180° around the line of reflection. The group might also note that a second reflection through the same line returns all points to their original position; ask about this property if nobody mentions it. A precise (although somewhat opaque) definition would state that each point on the line l of reflection is mapped to itself, while any other point P is mapped to the unique point P' such that l is the perpendicular bisector of $\overline{PP'}$.

In order to gain further intuition for reflections, draw a few geometric objects to one side of the line, then ask students what aspects of these figures will remain unchanged after they are reflected over the line. (Reflections preserve lengths, angle measures, areas, and practically any other measurable quantity. In short, the image objects will be congruent to the originals.) Then ask what features of the figures will be different after reflecting them. (Their location in the plane will change, of course. More subtly, their orientation will be reversed. Thus if vertices A , B , and C are labeled in clockwise order around a triangle, then their images A' , B' , and C' will appear in counter-clockwise order. This produces the “mirror-image” effect.)

Once participants have a solid grasp of reflection in the plane, have them consider the surface of a sphere instead. How could we perform a reflection on this surface? They should realize that one can reflect over a great circle in this case. So draw a horizontal great circle (i.e. an equator) and inquire as to what precisely it means to reflect the surface of the sphere across this circle. (Points on the equator remain fixed, while pairs of points directly above and below one another swap places.) Then have students rattle off properties of this sort of reflection. They should conclude that qualitatively there is very little difference between reflections in circles and in lines, except for the surfaces on which the action takes place.

Just for fun, announce that the surface of a sphere has exactly one more point than a plane. (Of course, this statement is not true, technically speaking.) To illustrate, place the large ball onto a handy table and have students imagine that a point light source is located at the north pole. If the ball were transparent, then light rays would pass through points on the ball and travel onwards until they struck the table. In this manner every point on the sphere except for the north pole N corresponds to a point in the plane tangent to the south pole S .² This supports our somewhat preposterous claim and, more importantly, introduces students to the concept of stereographic projection. Have students figure out what happens to a line in the plane passing through S when it is projected up onto the sphere. (It becomes a line of longitude; i.e. a great circle passing through N and S .) Then let them guess what happens to a line in the plane not passing through S . (It becomes a circle through N , but not S .) Finally, they should conjecture (correctly) that the image of a circle in the plane is a circle on the sphere not passing through N . Students are now prepared to blur the distinction between lines and circles and accept the notion of the inversive plane, which has one extra point included at “ N -finity.”



²Image from <http://aleph0.clarku.edu/djoyce/java/compass/compass4.html>

$$\frac{NS}{SQ} = \frac{SP}{NS} \implies (SP)(SQ) = (NS)^2 = (SV)^2,$$

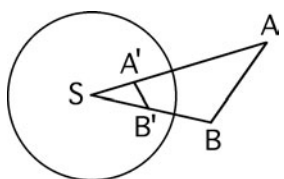
Allow students a few minutes to experiment with the effect of inversion upon various lines and circles. For instance, a circle with center S and radius $2r$ becomes a circle with center S and radius $r/2$. They should also conclude that lines through S

stay put, although points move around within the line. In order for this to work out neatly, lines must be thought of as including point N , making them topologically equivalent to circles. Help the group realize that lines not passing through S become circles through S and conversely. Finally, they should decide that circles not passing through S transform into other circles not containing S . If we adopt the convention that lines are really just circles passing through the point at infinity, then the foregoing discussion can be nicely summarized by stating that in the inversive plane an inversion maps circles to circles.

With this background in place the discussion might continue in several possible directions. One could prove a couple of the conjectures just made, such as the fact that lines not containing S are inverted to circles through S . Alternatively, one could develop a formula that indicates how distances are affected by an inversion, from which the “proper” derivation of Ptolemy’s Theorem quickly follows. Another nice exploration involves finding circles which remain fixed under inversion, which permits the solution of a particularly spectacular geometry problem. Each of these ideas is fleshed out below, and more can be found in the further problems. Mix and match them to create a presentation that is most suitable to the audience.

It is natural to seek explanations for some of the conjectures made regarding the fate of lines and circles under inversion. Begin by making sure that everyone understands why lines through S remain fixed, even though the individual points of the line move around. It helps to mark four special points: S , N , and the two points where the line meets the circle of inversion. These four points divide the line into four pieces. It is then a simple matter to figure out where each portion of the line goes under the inversion. Next, it is straight-forward to confirm that circles with center S turn into other circles with center S by appealing directly to the definition. Of course, the circle of inversion is mapped to itself.

It is less obvious why lines not containing S are transformed into circles through S . It will be helpful to consider the effect of inverting two points A and B , which we place outside the circle of inversion for greater clarity. Draw their images A' and B' along



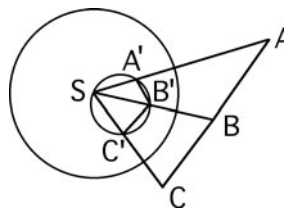
with the circle of inversion and its center S , but don't include any of the segments in the diagram yet. Ask students what they already know about the picture. Responses might include the fact that S , A' , and A are collinear in that order, and similarly for S , B' , and B . At this point draw segments $\overline{SA}$ and $\overline{SB}$. We also know that $(SA)(SA') = r^2$ and $(SB)(SB') = r^2$ by definition of inversion. With luck, a sharp shooter will suggest that $\triangle SAB \sim \triangle SB'A'$; otherwise opine that the key to further progress will again be a pair of similar triangles. Once they are identified, with vertices listed in the correct order, it is fair to draw in segments $\overline{AB}$ and $\overline{A'B'}$ as well. Finally, students can prove similarity by observing that both triangles share the angle at S and have adjacent sides in the same proportion; this follows from the fact that $(SA)(SA') = (SB)(SB')$.

Our reward for these efforts is the knowledge that $\angle SAB \cong \angle SB'A'$ and that $\angle SBA \cong \angle SA'B'$. For now suppose that we take three points on a line, which we again place outside the circle of inversion. Draw the diagram shown below, and inquire how we might prove that the images A' , B' , and C' , along with S , all lie on a single circle. (The congruent angles are the key. We have just seen that $\angle SAB \cong \angle SB'A'$

and $\angle SCB \cong \angle SB'C'$. Therefore

$$m\angle A'B'C' = m\angle SAB + m\angle SCB = 180^\circ - m\angle A'SC'.$$

In other words, angles $\angle A'B'C'$ and $\angle A'SC'$ are supplementary, which proves that S , A' , B' , and C' are concyclic.) So are we done? Not quite—we must show that the remaining points on the line also lie on this circle. So let D be any other point on the line, and ask why we can conclude that D' also lies on the inner circle. (There is a unique circle passing through S , A' , and B' . We have already seen that C' lies on this circle. By the same argument we know that D' is also located there.) When the proof is complete, invite students to investigate how the steps change when the line intersects the circle. Then challenge students to use similar reasoning to prove that circles not containing S are inverted to other circles not passing through S . Since this problem is significantly more difficult, one might offer a small prize for any student who submits a complete solution (or close enough) either during the math circle or later in the week.



It is entertaining to consider how lengths are affected by an inversion. For example, suppose that we designate a circle of inversion with radius 1 and that we select points A and B for which $AB = 4$. Let students figure where to locate these points so that their images A' and B' satisfy $A'B' < .01$, or $A'B' > 100$, or $A'B' = 4$. (Placing both A' and B' sufficiently far from S will give the first inequality, while placing A very close to S will satisfy the second inequality. One clever method to ensure that $A'B' = 4$ consists of letting $SA = 4$ and $SB = 1/4$, then opening angle $\angle ASB$ wide enough so that $AB = 4$.) The moral of the story is that distance $A'B'$ clearly depends on how far A and B are from S . To obtain a precise formula, return to the paragraph above in which students are led to discover that $\triangle SAB \sim \triangle SB'A'$. After they have proven this fact, have them derive an expression for $A'B'$ in terms of distances in the original diagram. (I.e. not involving A' or B' .) They should find that

$$\frac{A'B'}{AB} = \frac{SA'}{SB} \implies A'B' = AB \cdot \frac{SA'}{SB} = AB \cdot \frac{r^2}{(SA)(SB)},$$

where we used the fact that $(SA)(SA') = r^2$ in the final step. It is interesting to note that although $A'B'$ depends on SA and SB , as expected, it is independent of $\angle ASB$.

There is an immediate, pleasing application of this formula to proving Ptolemy's Theorem. To motivate this result, first plot three points A , B , and C in the plane, then ask students how one can test whether points A , B , and C are situated on a single line based solely on the knowledge of distances AB , AC , and BC . (The three points are collinear if and only if two of the distances sum to the third. In fact, the distances even determine the order of the points; for example, if $AC + BC = AB$ then C lies between A and B , on segment $\overline{AB}$. This is the triangle inequality, of course.) Now ask whether there is an analogous test for circles. In other words, given four points A , B , C , and D , can we ascertain whether they are concyclic given only the distances between each pair of points? Furthermore, is it possible to predict the order of the points around the circle? It is satisfying to learn that we can answer both questions in the affirmative.

Before searching for such a test, it is necessary to underscore how to apply inversion strategically to geometry problems. First, this technique is primarily applicable to problems involving circles. And secondly, the main advantage comes from the ability to transform circles into lines. Just to review, draw a circle and inquire which inversions will turn this particular circle into a line—any circle of inversion whose center lies on the given circle will do the job.

Now plot points A , B , C , and D in the plane and have the group brainstorm as to which inversions might be most practical. (Use a circle of inversion with center at one of the given points. As we shall see, its radius doesn't matter, although choosing D for its center is more convenient for notation.) Draw several possible configurations for A' , B' , and C' depending on whether or not the original four points lay on a single circle, and if so, in what order. For example, they should conclude that the original points are concyclic in alphabetical order if and only if B' lies on segment $A'C'$. But we already know that this occurs precisely when $A'B' + B'C' = A'C'$. Let students figure out where to go from here. (Use the inversion distance formula.) Hence this equality is equivalent to

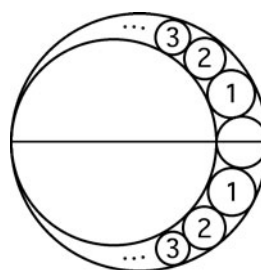
$$AB \cdot \frac{r^2}{(DA)(DB)} + BC \cdot \frac{r^2}{(DB)(DC)} = AC \cdot \frac{r^2}{(DA)(DC)}.$$

Cancelling the r^2 term and clearing fractions yields

$$(AB)(CD) + (AD)(BC) = (AC)(BD).$$

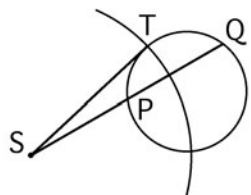
Of course, if B' does not lie on $\overline{A'C'}$ then all the $=$ signs should be replaced with $>$ signs. Similar considerations lead to the full statement of Ptolemy's Theorem; namely, that given any four points in the plane then the sum of any two of the quantities $(AB)(CD)$, $(AD)(BC)$, and $(AC)(BD)$ will be greater than or equal to the third, with equality if and only if the four points lie on a circle. In case of equality, the order of the points around the circle is determined by examining the largest product; the group can figure out precisely how this can be done.

The most satisfying applications of inversion are those which unravel hopelessly complicated problems by transforming them into transparently clear diagrams. Such is the case with the following beauty, which appeared as a cover problem from the now discontinued journal *Arbelos*. In the diagram at right, a large circle is given with a horizontal diameter. Two other circles are drawn with centers on this diameter so that all three are mutually tangent. A series of smaller circles are then drawn in the remaining space as follows. The first pair (labelled with a 1) are constructed tangent to each of the initial three circles. The ensuing pairs are positioned so that they are tangent to two of the original circles and to one of the circles from the previous pair. The task is to prove that the distance between the centers of the n^{th} pair of circles is equal to precisely $2n$ times the length of their diameter.



It is natural to feel uneasy about utilizing inversion since the statement to be proved involves several lengths and we have seen how thoroughly inversion can mangle distances. Therefore it should come as a relief to learn that there is a handy technique for setting up an inversion in such a way that certain circles in the plane remain fixed

in place. Lead participants to discover this technique by drawing a circle in another part of the board along with a point S some ways outside the circle. Then inquire whether it is possible to choose just the right radius r so that inverting about a circle



with center S and radius r will leave the given circle fixed. Have them imagine how the points on the circle would have to move about for this to be possible—recall that points swap places along rays emanating from S . After a few moments, focus their attention on the extreme case; namely, one of the rays out of S which is tangent to the circle. Eventually insight will strike and some participant will realize that the point of tangency must be mapped to itself, otherwise it will not return to the circle. This implies that the distance from S to the point of tangency T is the only viable option for the desired radius r . From here it is not hard to prove that all other pairs of points do exchange places in the hoped for manner; use the Power of a Point theorem or show that $\triangle STP \sim \triangle SQT$ to conclude that $(SP)(SQ) = (ST)^2 = r^2$, implying that P is mapped to Q and conversely. Because this pair of circles intersects at right angles, they are known as orthogonal circles.

The group is now prepared to polish off the pretty problem posed previously. Focus their attention on the second pair of circles (labelled with a 2) and ask which inversion is likely to be the most useful. If they have not already been informed, remind them that the main advantage comes from the ability to transform circles into lines by selecting a circle of inversion whose center lies on one or more circles in the diagram. Let students search for the best inversion on their own before broadcasting ideas; faces will light up as they discover it. After a suitable interval allow someone the honor of presenting the solution: choose a circle of inversion whose center is the leftmost point in the diagram (the point of tangency of the two largest circles) and whose radius ensures that both circles numbered 2 will remain fixed. Upon inverting the two large circles turn into parallel vertical lines (be sure to explain why) sandwiching the chain of smaller circles between them. The smaller circles are all tangent to their neighbors and to the parallel lines, hence they are all congruent; in particular, they are all the same size as the circles numbered 2 *before the inversion*. It is now perfectly clear that the distance between the centers will be four times their diameter; just count the intervening circles! A similar argument establishes the claim for the n^{th} pair of circles in general. It is hard to think of a nicer point on which to end the math circle.

Further Problems.

1. How can we obtain an ordinary reflection in the plane over a line as the stereographic projection of a reflection in a circle on the surface of a sphere?
2. Suppose we take a circle on a sphere other than a great circle. (Such as a line of latitude above the equator.) How could we define “reflection in a circle” in this case? What happens to circles under this sort of reflection?
3. We have shown that a circle through the center S of inversion becomes a line. Demonstrate that the diameter of the original circle having S as an endpoint is perpendicular to the resulting line.
4. Prove that a circle not passing through S inverts to another circle.
5. Let A , B , C , and D be four points in the given order around a circle with center O . The circumcircles of triangles AOC and BOD intersect a second time at K . Prove that $(AK)(KC) = (BK)(KD)$.

There is a further property enjoyed by inversion which has not yet been mentioned; namely, that angles are preserved by inversion. In particular, if two lines or circles intersect at a particular angle, then their images will also meet in the same angle. (To measure the angle at which a circle meets another circle, draw the tangents at the point of intersection.)

6. Given three circles passing through a common point P and meeting at 60° angles, let X , Y , and Z be the points where these circles meet a second time. Prove that $(PX)(YZ) = (PY)(XZ) = (PZ)(XY)$. (To obtain such a diagram, draw three lines through a given point P all meeting in 60° angles, then draw a circle of any size tangent to each line at P .)
7. Suppose that four circles are given, each externally tangent to its two neighbors. Prove that the four points of tangency themselves lie on a circle.
8. Given three mutually tangent circles, construct the small circle in the space between them that is externally tangent to all three. Then modify your approach to obtain the large circle around them that is internally tangent to all three.
9. Suppose that two non-intersecting circles are given, one contained in the interior of the other. If the two circles are not already concentric, prove that there exists an inversion which maps them to a pair of concentric circles.
10. (Steiner's Porism) Given a pair of non-intersecting circles, one contained inside the other, construct a chain of circles as follows. Draw any circle in the region between the original pair that is tangent to both of them. Continue inserting circles into the chain, each circle tangent to the previous one as well as to the original circles, until some circle intersects the initial circle in the chain. Suppose that the final circle happens to be tangent to the initial one, thereby completing the chain of tangent circles. Prove that in this case it will be possible to complete the chain regardless of the position of the initial circle in the chain. (Nice Java applets illustrating this phenomenon can be found on the internet.)

Hints and Answers.

1. Instead of reflecting across the equator, use the prime meridian or any other great circle passing through the north and south poles.
2. Consider the cone which is tangent to the sphere along the given circle. Lines emanating from the vertex of the cone and passing through its interior will intersect the sphere twice; declare that these pairs of points trade places when reflected through the circle. With this definition, circles on the surface of the sphere are still mapped to circles, although it is no longer obvious why.
3. Observe that points further from S in the original diagram are inverted to points closer to S . Let D be the other endpoint of the diameter through S . Since D is furthest from S on the circle, D' will be the point of the image line nearest to S , which implies that line SD' (which contains D) is perpendicular to the image line.
4. A direct geometric argument is slightly tricky to set up and involves several cases. For example, suppose that A , B , C , and D appear in this order around "the far side" of a circle from S , the center of inversion. By inscribed angles we know that $\angle ABD \cong \angle ACD$, thus $m\angle SBA + m\angle SBD = m\angle SCA + m\angle SCD$. The congruent angles found earlier then imply that $m\angle SA'B' + m\angle SD'B' = m\angle SA'C' + m\angle SD'C'$.

A little angle chasing reveals that $\angle A'B'D' \cong \angle A'C'D'$, from which it follows that the four image points are also concyclic.

5. Perform an inversion in the given circle, so that $A' = A$ and so on. The circum-circles become lines intersecting at K' ; applying the Power of a Point theorem gives $(A'K')(K'C') = (B'K')(K'D')$. Finally, use the distance formula to convert back to distances in the original diagram.

6. An inversion with center P will transform the three circles into three lines which still meet in 60° angles. Hence $X'Y'Z'$ will be an equilateral triangle, meaning that $X'Y' = X'Z' = Y'Z'$. Converting back to unprimed lengths yields the desired equalities.

7. Label the four points of tangency as A, B, C , and D in order along the chain of four tangent circles. Then an inversion with center D yields a new diagram in which two circles are externally tangent to one another at B' as well as tangent to one of a pair of parallel lines at A' and C' . It suffices to show that A', B' , and C' are collinear. This can be done using a pair of similar triangles, or by considering a dilation (involving a 180° rotation) about B' which maps one circle into the other.

8. Let S be one of the points of tangency and perform an inversion with center S which leaves the third circle (not containing S) fixed. The first two circles become a pair of parallel lines tangent to the third circle and perpendicular to the segment joining the centers of the first two circles. The desired inscribed circle becomes a congruent circle tangent to both lines and the third circle. It is now straight-forward to construct the point of tangency of the two congruent circles and then map it back along the ray towards S to find the location of the desired point of tangency in the original diagram. Once all three points of tangency are pinpointed the inscribed circle can be constructed.

9. Clearly the center of inversion should be located on the line joining the centers of the two given circles. There are clever purely geometric constructions for finding such an inversion, although an algebraic approach is more direct. The solution is not unique, and the point of the exercise is primarily for use in the ensuing problem.

10. Perform an inversion which carries the two initial circles to a pair of concentric circles, if they are not already concentric. The assertion now becomes obvious.

Part III

Appendices

Appendix A

Sample Documents

The following pages contain a variety of forms and other documents related to the administration of a math circle. Most of them have been reformatted to fit into the available space, but they are otherwise accurate reproductions of the originals. The author makes no attempt to disguise the fact that these are primarily taken from his own Stanford Math Circle files, and as such are not representative of the full range of documents in use at other circles. The intent, rather, is to provide a sampling of documents which can serve as a springboard for organizers as they set about assembling their own set of forms.

A copy of this sheet is set out each week on the main desk outside the classroom door at the Stanford Math Circle. The information gathered is all useful, although the ability to contact students directly via email is the most valuable aspect of the registration process. Convincing students to print their email address legibly is also the greatest challenge in gathering the information.

[illegible]

Attendance Sheet

[illegible]

Welcome Letter

In lieu of holding a meeting with parents, the Stanford Math Circle opts to set out a letter addressed to parents for the first several meetings of each term. The letters are also sent home with students, since many of them are simply dropped off at the door of the building.

Dear parents,

Welcome to the Stanford Math Circle! We are excited to start up again for our second year. There are many great speakers and topics lined up for the twenty-four sessions scheduled during the 2006-2007 academic year. Almost all of the information related to this math circle can be found on our web site, at <http://www.stanfordmathcircle.org>, including the schedule for the year (which will be updated at regular intervals), directions to our classroom, and documents related to past meetings in case you happen to miss one.

As you may know, we operate solely through parental support and donations rather than charging tuition for classes offered. Funds raised will be applied exclusively towards operating costs, speaker honorariums, food, and prizes. For families wishing to participate, we suggest a \$40 donation. (Since we have a balance left over from last year, families that helped out in the past should not feel obligated to make another contribution.) At the end of each session (Fall 2006 and Winter 2007) we will post a statement of accounting at the web site to document how funds have been used. All donations are tax-deductible; just make checks payable to “Art of Problem Solving Foundation” and write “Stanford Math Circle” on the memo line. *Please do not make checks payable to Stanford math Circle.* Better yet, contribute online—just visit our web site and follow the directions.

There is one other important way that you can become involved in your friendly neighborhood math circle. I have provided light snacks for our circle today, but I only plan to do so for the first math circle of each session. So we are in need of a volunteer who is willing to coordinate the refreshment effort this fall, and other parents who can bring food on subsequent Sundays. To get the ball rolling, I have put together a sign-up sheet which is available at the registration desk. If you are able to help, please choose a date when you could provide snacks for around forty students and fill in the corresponding line.

Thanks again for joining us today. If you have any ideas, questions, or concerns, please do not hesitate to contact me at samv@stanfordmathcircle.org. I hope that everyone has a great time doing great mathematics.

Sincerely,

Sam Vandervelde

Coordinator,
Stanford Math Circle

Snack Sign-Up

The wonders of snack time have already been sufficiently proclaimed. Here is one way to help make them happen. A copy of this sheet (once it is filled out) can be given to the parent volunteer who oversees snack time.



STANFORD
MATH
CIRCLE

Volunteer for Snacks!

There's nothing like a few munchies to make for an exciting math circle. Please consider becoming involved this winter by volunteering to bring food for one of the Sundays. When planning on items, please keep the golden rule of snacks in mind: "LESS MESS IS BEST." For example, cookies, fruit, and popcorn would probably work well. Cake is definitely bad. Popsicles are probably not the best idea either. Please do not bring beverages – there are water fountains in the building which should suffice. All food should be set up and consumed in the hallways or common areas. *Please do not bring food into the classrooms.* Thanks!

Winter Session 2007

Date	Name	Phone Number	Email Address
Jan 14			
Jan 21			
Jan 28			
Feb 4			
Feb 11			
Feb 18			
Feb 25			
Mar 4			
Mar 11			
Mar 18			
Mar 25			
Apr 1			

Email for Invited Speakers

Dear Leonhard Euler,

Thanks for agreeing to lead a couple of math circles on the occasion of your 300th birthday. We have a group of around twenty-five students participating on a consistent basis. They are mostly high school kids, which is our target audience, but we get some middle schoolers and (of course) parents as well. I've advertised that the Stanford Math Circle is about "Having a great time learning great mathematics," so I hope that you will enjoy presenting some mathematical topic that you particularly like. My biggest piece of advice is to develop your ideas at a reasonable pace, but not to shy away from substantial mathematics. Of course, please include lots of fun exercises for the kids to try out along the way. A number of them are quite sharp, but the majority are there because they love the subject, not because they are going to ace the next AIME.

OK, on to the details. You will find lots of useful information at our web site <http://www.stanfordmathcircle.org>, including driving directions, how to find our room, the schedule, and more. The circle takes place from 2:00 to 4:00 on Sunday afternoons in room 380-C in the basement of Sloan Hall on Stanford campus. Depending on how much time you need to set up, you might plan on arriving around 1:50 or so. Let me know what sort of equipment you might need; the room is equipped with a chalkboard, an overhead projector, and a ceiling-mounted projector for laptops, VCR tapes, and DVDs.

I will begin promptly at 2:00 with some warmup activities and any announcements. I will then introduce you at 2:15. You will have the remainder of the time, except for a ten minute break for snacks halfway through, somewhere around 3:00. (So you can plan on around 90 minutes of time for speaking.) Everyone has their own style of presenting, but it helps to keep in mind the "math circle philosophy" as you prepare your talk: the idea is to employ a dynamic, interactive, problem-solving approach to math. Ideally you will strike a good balance between presenting material and having them try it out for themselves. (In other words, avoid lecturing for stretches of more than five minutes at a time.) It's a lot of fun.

I would like to make a worksheet of practice problems based on your talk available at the web site. I typically type up a worksheet afterwards containing the questions discussed during the circle (or small variations of them), along with a few extras for practice, and then post this at the web site. This has the advantage of helping me to hold their attention during the actual circle, but the disadvantage that fewer kids actually wind up working on the extra problems, I suspect. So if you prefer, you can have a worksheet ready for them at the start of the circle; I will make enough copies and then distribute them whenever you like (start of circle, during break, or on their way out). Regardless, if you can email me a file by that Sunday (LaTeX, PDF, Word doc, whatever), then I will post it by the evening.

You are more than welcome to visit our circle one of the Sundays prior to your presentation to get a sense of what the math circle is all about. In fact, if this is your first opportunity to lead a math circle I would highly recommend it! In addition, because of the generous support of our parents, I am please to be able to offer you a modest \$150 honorarium as thanks for giving two talks at the Stanford Math Circle. If you could let me know the proper mailing address, then I will arrange for a check to be sent shortly after your final talk. Thanks!

Those three little dots
Sam Vandervelde
October 22, 2006

The expressions below are arranged in no particular order. Choose one that looks interesting and decide how its value may be precisely defined as the limit of a sequence. Then try to discover the exact value by hand, checking your answer by computing a good numerical approximation. Finally, for the stout of heart, prove that your answer is correct. You are welcome to check any of your answers or proofs—just send them to me at csaav@math.stanford.edu.

$$\sqrt[3]{2207 - \frac{1}{2207 - \frac{1}{2207 - \dots}}}$$

$$2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3 + \frac{3}{4 + \dots}}}}$$

$\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\dots$

$$\frac{1}{1} - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \dots$$

$$\sqrt{3\sqrt{9\sqrt{27\sqrt{81\cdots}}}}$$

$$\frac{1}{1} + \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{5}{120} + \cdots + \frac{F_n}{n!} + \cdots$$

$$\sqrt{90 - \sqrt{90 - \sqrt{90 - \sqrt{90 - \dots}}}}$$

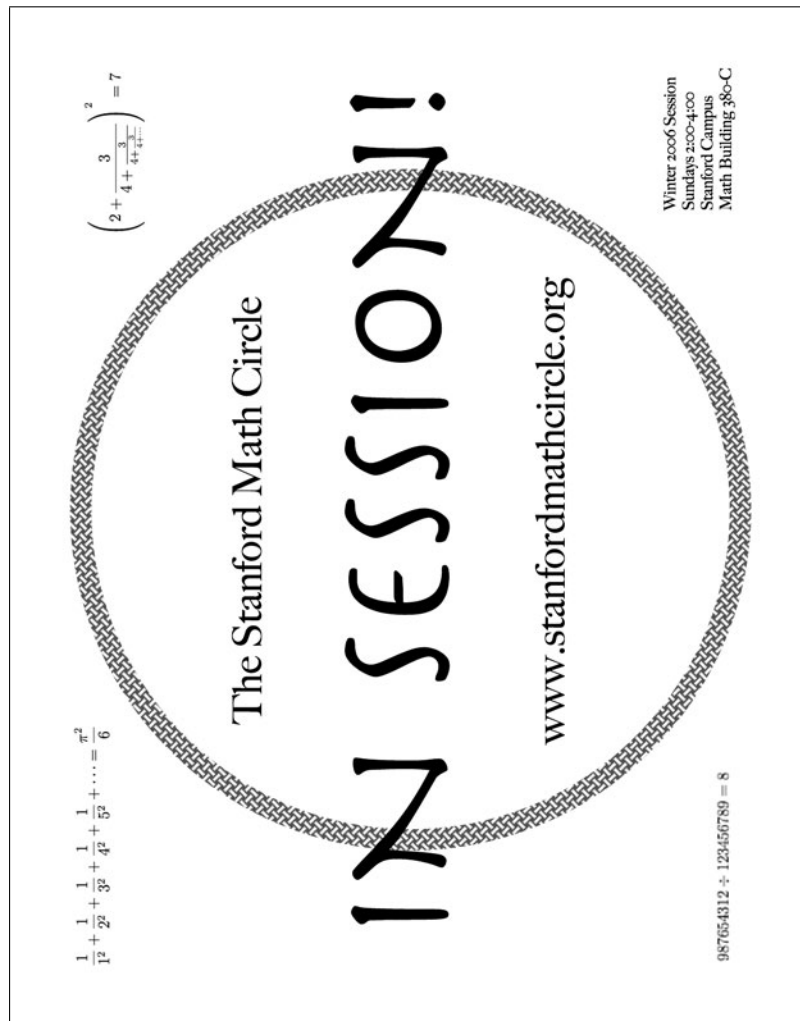
$$2 + \frac{3}{4 + \frac{3}{4 + \frac{3}{4 + \dots}}}$$

$$1 + \frac{3 + \frac{1 + \frac{3}{2} + \dots}{3 + \frac{1}{3} + \dots}}{2 + \frac{2 + \frac{1}{2} + \dots}{1 + \frac{3}{2} + \dots}}$$

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\dots}}}}$$

Math Circle Flyer

Following the December holidays this flyer was sent out to all students on the email list as an attached pdf file. The email described the upcoming sessions, invited students to return to enjoy more scintillating sessions, and suggested that they forward the flyer to any friends in the area whom they thought might be interested in attending.



End of Year Survey

Prior to the pizza party at the final session all students were asked to provide anonymous feedback on their experience at the Stanford Math Circle. This survey is relatively brief, but it does touch on all aspects of the circle which the organizers felt were important. A coordinator wishing to assemble data for a grant report might want a more detailed survey; an example of one appears in the supplementary grant report described by a later appendix.

STANFORD MATH CIRCLE

MINI-SURVEY AND PIZZA PARTY TICKET

To help us plan next year, we would appreciate your kind yet frank appraisal of your friendly neighborhood math circle.

1. Overall, was the choice of topics appropriate?

too familiar *just right* *too obscure*
1 2 3 4 5 6 7

2. On the whole, how was the level of material presented?

piece of cake *just right* *way hard*
1 2 3 4 5 6 7

3. Did the speakers make the mathematics exciting?

ho-hum *not bad* *inspiring*
1 2 3 4 5 6 7

4. What did we do well that we should keep for next year?

5. What didn't go so well? How should we change?

Appendix B

Warm-ups

At the request of the San Diego and San Jose Math Circles, the author is including this section outlining the warm-ups used recently at the Stanford Math Circle. A warm-up is a mathematical morsel in the form of a clever puzzle, a small but fascinating problem, or an appealing little result. Each should last eight to ten minutes, and has a two-fold purpose. Naturally a warm-up should be mathematically catchy; the idea is to quickly capture students' attention and engage their minds so that they are ready to hit the ground running when the main talk begins. But they also provide a means for starting the math circle at the advertised time while still allowing some built in slack for students who arrive a few minutes late, so that the audience is as complete as possible when the talk commences.

The items below are only sketched briefly, leaving the entertaining task of working out the details to the coordinator and students. They are primarily suitable for high school students, but many can be adapted for younger audiences as well. Have fun with them!

1. "Sums and Cubes I." Find three integers x , y , and z such that $x + y + z = 3$ and $x^3 + y^3 + z^3 = 3$. (The obvious answer is $x = y = z = 1$.) Now find a second solution. (The astonishing fact is that $x = y = 4$, $z = -5$ also works!) Next define a *beastly number* as a positive integer which is equal to the sum of its digits plus the sum of the cubes of its digits. Demonstrate that 152 is not beastly, as an example. Announce that there are precisely six beastly numbers and challenge kids to find as many as they can. (The answers are 12, 30, 666 of course, 870, 960, and 1998.)

2. "Sums and Cubes II." Ask if anyone is familiar with the formula for the sum of the first n cubes. Present the fact that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

Challenge the group to find a proof by picture of this fact. As a hint, help them realize that 3^3 can be thought of as the total area of three 3×3 squares. As a further hint, have them try fitting a 1×1 square, two 2×2 squares, and three 3×3 squares together to make a 6×6 square. (The 2×2 square has to be chopped up slightly to fit.) Continue trying examples until everyone is convinced of the above identity.

3. “Sums and Cubes III.” Present the following gem, due to Ross Honsberger. Announce that the first n positive integers are not the only set of numbers with the property that the sum of their cubes is equal to the square of their sum. Have students choose any two-digit number N , preferably not prime. Then list all the positive divisors of their number, including 1 and N . Finally, underneath each number in this list tally how many positive divisors it has, thereby creating a new list of equal length. Then this new list has the desired property. For example, the divisors of 6 are 1, 2, 3, and 6; which have 1, 2, 2, and 4 positive divisors, respectively. Sure enough, $1^3 + 2^3 + 2^3 + 4^3 = 81 = (1 + 2 + 2 + 4)^2$.

4. “Efron’s Dice.” Describe four dice having the following numbers on their faces.

A 0, 0, 4, 4, 4, 4

B 3, 3, 3, 3, 3, 3

D 1, 1, 1, 5, 5, 5

C 2, 2, 2, 2, 6, 6

Then imagine a game in which two players each pick a die and roll it once, the winner being the person with the higher number. Compute the probability of winning in the games A vs B, B vs C, C vs D, and D vs A. Surprisingly, the first die in each pairing wins with probability $2/3$.

5. “The Game of 24.” The goal in each of the following cases is to produce the fifth number using each of the first four numbers exactly once, along with the symbols $+$, $-$, $\cdot$, $/$, and parentheses used as desired. The puzzles become increasingly tricky. Answers are given to the right, often more than one answer is possible.

- | | | |
|----|-----------------------------------|--------------------------------|
| a. | 1, 4, 7, 10 $\longrightarrow$ 13 | $(10 + 7 - 4) \cdot 1 = 13$ |
| b. | 2, 3, 5, 8 $\longrightarrow$ 13 | $(3 - 2) \cdot (5 + 8) = 13$ |
| c. | 1, 2, 4, 8 $\longrightarrow$ 16 | $(8 \cdot 4)/(2 \cdot 1) = 16$ |
| d. | 3, 5, 8, 13 $\longrightarrow$ 1 | $5 \cdot 8 - 3 \cdot 13 = 1$ |
| e. | 2, 5, 11, 17 $\longrightarrow$ 37 | $(5 \cdot 17 - 11)/2 = 37$ |
| f. | 1, 5, 5, 5 $\longrightarrow$ 24 | $(5 - 1/5) \cdot 5 = 24$ |

6. “Numbers Beginning With P” Think of as many as possible—the list might include primes, perfect numbers, perfect squares or cubes, powers, palindromes, pentagonal numbers, pi, or Planck’s constant. Then draw a 3×3 crossword grid and supply the following clues for the six three-digit numbers that appear there.

ACROSS

DOWN

1. Prime

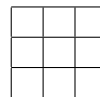
1. Power of 3

2. Perfect square

2. Perfect square

3. Perfect cube

3. Prime



As a bonus, there will be two palindromes in the completed puzzle. (Answers are 277, 484, 343, 243, 784, and 743, respectively.)

7. “Points of Intersection” The goal is to position a prescribed number of lines in the (Euclidean) plane to obtain exactly 100 points of intersection. Begin with 100 lines, then try 29 lines, 21 lines, 19 lines, and finally 15 lines. For example, by arranging for 99 lines to be concurrent at a single point with one other line which intersects all of these we can create 100 points of intersection with 100 lines. The other four examples are all possible, but the last couple require some determination.

8. “A Ladder Locus” A ladder, originally upright against a vertical wall, slides downward and outward, always in contact with the wall, until it crashes to the ground.

What figure is traced out by a hapless painter positioned midway up the ladder? Students can conjecture that the answer is a quarter of a circle with radius equal to half the length of the ladder, centered at the point where the wall meets the ground. Then have students come up with variations on this problem. (Consider positions other than the midpoint of the ladder, or try changing the angle at which the wall meets the ground. In both cases ellipses result.) Figure out how to rephrase the question so that the answer is an entire circle instead of just a quarter circle.

9. “Mix and Match” Draw a picture of an open ray and a closed ray, such as the sets of points $(x, 0)$ with $x \geq 0$ and $(x, 1)$ for $x > 0$. Ask students which object contains more points. Then challenge them to find an explicit bijection between the two sets. (Map $(x, 0)$ to $(x, 1)$ whenever x is not an integer, otherwise map $(x, 0)$ to $(x + 1, 1)$.) Gradually include more objects, if desired. For example, find a bijection between a closed ray and a line, or between an open ray and an open segment.

10. “Beguiling Tilings I” Figure out how to tile a quadrilateral region with segments. In other words, cover the entire area with non-intersecting segments, each of which has positive length. (No points may be used, and segments cannot overlap at their endpoints.) Start with a trapezoid, then try a convex quadrilateral. A triangular region is more tricky—if needed hint that one of the segments will be a side of the triangle, while another will extend from the opposite vertex to the midpoint of one of the remaining sides. This now makes it possible to tile a concave quadrilateral region. As a bonus challenge, try tiling a circular disc.

11. “Beguiling Tilings II” Demonstrate how to tile all of space with non-degenerate circles. To help set the stage, point out that this cannot be done for the plane. (But one can come close; removing a single point makes it possible.) Here is a beautiful method for accomplishing the task. First convince everyone that the surface of a sphere missing two points can be tiled with circles: consider the two planes tangent at the missing points, then rotate a third plane passing through their line of intersection. Next draw unit circles in the xy plane centered at the points $(n, 0, 0)$ for all integers $n \equiv 1 \pmod{4}$. Finally, argue that any sphere centered at the origin intersects this family of circles at precisely two points, and put it all together.

12. “Beguiling Tilings III” Direct the group to tile an open segment, say $0 < x < 1$, with closed segments of positive length. Students will almost inevitably settle upon a Cantor-like construction in which they successively cover the middle third of each remaining open interval. (If other lengths are proposed, suggest using thirds to keep things more symmetric and simple.) Debate whether or not every point is included in this process. As a check, compute the total length of the segments used in the tiling, namely

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = 1.$$

Turn the tables by observing that the number $\frac{1}{4}$ is left untouched at every stage, since

$$\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots = \frac{1}{4}.$$

13. “The Digit Puzzles” Each of the brain-teasers below involves the digits from 1 to n , where n is a digit between 1 and 9, inclusive. Two or three of them are about right for a warm-up. Answers appear below the list of questions.

PUZZLE 1: Using exactly four 1's, make a perfect square. How many different solutions are possible?

PUZZLE 2: Place a standard mathematical symbol between the digits 1 and 2 to create a number that lies strictly between 1 and 2. Repeat this puzzle, except now obtain a rational number between 1 and 2.

PUZZLE 3: Write down as large a number as possible using only the digits 1, 2, and 3 but no other mathematical symbols.

PUZZLE 4: Name as large a perfect square as possible which contains only the digits 1, 2, 3, or 4.

PUZZLE 5: Using the digits from 1 to 5, create a five digit number which is divisible by at least five factors of 2.

PUZZLE 6: Create three positive integers using the digits from 1 to 6 so that the product of the two smaller numbers equals the larger. (Note that this can also be done with the digits from 1 to 4 or 1 to 5.)

PUZZLE 7: Find an efficient way to prove that when the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

is written as a fraction in lowest terms, the numerator will be divisible by 7^2 .

PUZZLE 8: Use each of the digits from 1 to 8 exactly once to form as long a list of primes as possible.

PUZZLE 9: A *digital number* is a nine-digit number containing each of the digits from 1 to 9 exactly once. Find a digital number that is a multiple of another digital number. (Hint: try to make their quotient as large as possible.)

Don't peek at the answers too quickly. When you are ready to check, here they are.

ANSWER 1: One can obtain 0, 1, 4, 9 (as $11 - 1 - 1$), 121, and of course $\square$. (Note the deliberately vague wording of the question!)

ANSWER 2: The first answer will probably be $1\sqrt{2} \approx 1.414$. To create a rational number instead, use a decimal point.

ANSWER 3: There's always 321. But 31^2 and 21^3 are bigger, while 2^{31} and 3^{21} are larger still. It is a good exercise to demonstrate (without resorting to a calculator) that 3^{21} triumphs.

ANSWER 4: Although there may be larger possibilities, one nice answer is $1111^2 = 1234321$.

ANSWER 5: The unique solution is 45312. It is not necessary to resort to trial and error; for example, begin by arguing that the final digit must be 2. Surprisingly, this number actually has eight factors of 2.

ANSWER 6: The only way to arrange the multiplication is $54 \cdot 3 = 162$. (We also have $13 \cdot 4 = 52$ and $3 \cdot 4 = 12$, with fewer digits.)

ANSWER 7: Add opposite pairs and then pull out a factor of $7/2$ to obtain

$$\frac{7}{2} \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{6} \right).$$

Next add the two outer terms, and the second factor of 7 becomes obvious.

ANSWER 8: No prime ends in 4, 6, or 8, so the longest possible list will contain five primes. This can be achieved by writing 2, 5, 41, 67, 83.

ANSWER 9: Remarkably, $987654312 = 8 \cdot 123456789$. As an extra challenge, prove that this phenomenon persists in other number bases.

14. “Fun with Fixed Points” Plot two points A and B on the board, then describe the following compositions of transformations. In each case there will be exactly one point in the plane which winds up back at its starting point; the challenge is to find the fixed point. (Recall that positive angles denote counterclockwise rotations. “Translate $A \rightarrow B$ ” refers to the translation which maps A to B . As a hint for the last several, draw a square grid which includes points A , B , and the midpoint of segment AB .)

- Translate $A \rightarrow B$, then rotate 180° about B .
- Translate $A \rightarrow B$, then rotate 180° about A .
- Translate $A \rightarrow B$, then dilate by a factor of 2 about B .
- Translate $A \rightarrow B$, then rotate 90° about A .
- Same as d. but in the other order.
- Rotate 90° about A , then rotate 90° about B .
- Same as f. but then also translate $B \rightarrow A$.
- Same as f. but then also translate $A \rightarrow B$.

ANSWERS Impose a coordinate system so that A is the origin and B has coordinates $(2, 0)$. Then the fixed point is...

- | | |
|--------------|--------------|
| a. $(1, 0)$ | e. $(1, 1)$ |
| b. $(-1, 0)$ | f. $(1, -1)$ |
| c. $(-2, 0)$ | g. $(0, -1)$ |
| d. $(-1, 1)$ | h. $(2, -1)$ |

15. “Name That Number” In this age of calculator use, many students have become adroit at recognizing familiar numbers in their decimal form. Give them a run for their money with this quiz.

- 3.162277660168379331998893544432718533720
- 2.718281828459045235360287471352662497757
- 1.618033988749894848204586834365638117720
- 1.570796326794896619231321691639751442099
- 0.707106781186547524400844362104849039284
- 496
- 1729
- 2520
- 40320
- 33554432

ANSWERS

- | | |
|--|---------------------------------|
| a. $\sqrt{10}$ | f. the third perfect number |
| b. e | g. $1^3 + 12^3$ or $9^3 + 10^3$ |
| c. $\frac{1}{2}(\sqrt{5} + 1)$, the golden mean | h. $LCM(1, 2, 3, \dots, 10)$ |
| d. $\pi/2$ | i. $8!$ |
| e. $\sqrt{2}/2$ or $\sin 45^\circ$ | j. 2^{25} |

Appendix C

Sample Grant Proposal

The document that follows constitutes the original successful grant proposal written by MSRI in September 2004 for the creation of the San Francisco Math Circle. It has been modified slightly so that the identity of the funding corporation remains anonymous. Note the main elements of the proposal and the order in which they appear:

- a brief description of the grant writer's institution and goals,
- a concise overview of the proposed program with clearly stated goals and means for achieving these goals,
- a short history of math circles and their impact,
- a complementary and slightly more detailed section explaining how the proposed math circle would fit into the existing mathematical environment and who would serve as coordinator,
- a plan for publicly recognizing the contribution of the sponsor,
- a proposed means of evaluating the success of the program,
- and an approximate budget.

Naturally every proposal will be unique; a few common sense principles will dictate how others might differ from the one presented below. For example, the length of the proposal will be roughly proportional to the scope of the program. And of course guidelines set forth by the funding agency would take precedence over the format shown here. However, this document does give a good idea of what goes into a successful proposal.

Mathematical Sciences Research Institute

Proposal to XYZ Corporation *for* San Francisco Math Circle Program

About MSRI

The Mathematical Sciences Research Institute (MSRI) exists to further mathematical research through broadly based programs in the mathematical sciences and closely related activities. MSRI conducts year-long and semester-long research programs in the mathematical sciences, both pure and applied. Each year, the Institute becomes the world's most important center of activity in the topics of its major programs. Introductory and specialized workshops and summer graduate programs complement and supplement these programs. The Institute is one of the major postdoctoral training sites in the mathematical world. Its education activities include programs with middle and high school students, and curriculum research. Its public outreach includes events in the "Conversations" series, often about popular plays and books. These public events include a recent event at the Herbst Theater with Steve Martin and Robin Williams conversing about math and science. From its beginning in 1982, MSRI has been the largest recipient of grant support from the Mathematics Division of the National Science Foundation, and receives additional support from other government agencies, private foundations, and academic and corporate Sponsors. Over 1700 mathematical scientists visit MSRI each year.

MSRI's goals are:

- To foster mathematical research by providing an environment in which mathematicians can work and collaborate, creating new mathematics and solving deep problems in the mathematical sciences;
- To bring together experts from different fields and strengthen the interaction of mathematics with its applications, stimulating mathematics with new problems and increasing the usefulness of mathematics in science and technology;
- To enhance the place of mathematics in our culture and increase public awareness of mathematics and its uses, so that it can continue to flourish and to attract the most able students.

Program Introduction

Mathematical Sciences Research Institute proposes the initiation of a Mathematics Circle in San Francisco to serve middle and high school students from San Francisco and Marin Counties. In 1998, MSRI initiated a program in Berkeley, California to identify and encourage middle and high school students interested in higher-level mathematics, the Berkeley Math Circle (BMC). There were over 50 students registered for the BMC last year, with about 30 students attending a 2-hour Sunday session at any particular time. We anticipate this same level of participation in the SFMC in its

first full year of implementation. The BMC and its accompanying program, the Bay Area Mathematics Olympiad (BAMO), provide interested and motivated middle and high school students with an opportunity to work on complex essay style mathematical problems. The concept of the San Francisco Math Circle is to serve a broader and younger student body, to develop their mathematical interest and talent. The purpose of the combined programs is to increase the quality and quantity of students who become mathematics educators and researchers, or who simply love and use mathematics in their studies, work and daily activities:

- To draw kids to mathematics and to motivate them to excel in the subject;
- To draw kids to the program who are economically disadvantaged and who might otherwise not attend;
- To prepare them for mathematical contests;
- To introduce them to the wonders of beautiful mathematical theories, and
- To encourage them to undertake futures linked with mathematics, whether as mathematicians, mathematics educators, scientists, computer scientists, economists or business leaders.

The San Francisco Math Circle (SFMC) will achieve these goals by:

- Exposing middle and high school students to exciting mathematics presented by research mathematicians, mathematics educators and internationally seasoned Olympiad problem-solvers;
- Providing extracurricular opportunities to extend mathematical knowledge and skill well beyond the high school curriculum;
- Creating a math fostering and friendly atmosphere at the math circle, where love for mathematics is the driving force behind devoting each Sunday afternoon to the SFMC sessions;
- Encouraging and facilitating teachers from schools with economically disadvantaged students to bring these students;
- Encouraging participation from economically disadvantaged schools by teacher incentives and by awarding the libraries of such schools with math books at the end of each semester for regular attendance in the circle and participation in BAMO.
- Giving opportunities to students to meet and work with other students of similar or superior math abilities and aspirations;
- Creating a forum (monthly contests) for testing one's math competition skills;
- Providing invaluable advice on college selection, summer math programs and internships, and future mathematical orientation.

History of Math Circles

Math circles originated in Hungary more than a century ago. They soon spread over Eastern Europe and Asia, and since then have produced many of the great scientists from those parts of the world, in mathematics and in other disciplines. The math circles also led eventually to the start of many national and international math contests, including the International Mathematical Olympiad (IMO) in 1959 in Romania. It is widely believed that it is the presence of these circles that has enabled the youth of

countries such as Russia, Bulgaria and Romania on average to outperform the United States at the IMO. Given the success of math circles in Russia and Eastern Europe, it is surprising that it has taken so long for the United States to develop similar programs. The Berkeley Math Circle was founded in 1998 to begin to correct this situation. Since then groups have begun in Palo Alto and San Jose. The San Francisco Bay Area was a natural choice for the site of math circles, both because of the large number of talented high school students in the area, and because of the proximity of world-class institutions such as MSRI from which experienced lecturers could be drawn.

Although it has existed for only a few years, the Berkeley Math Circle and the competition it organizes, the Bay Area Mathematical Olympiad, have become known to thousands of people in the San Francisco Bay Area and across the country, both through word-of-mouth and in the media. BMC frequently receives request from schools and universities across the U.S. for help in setting up their own circles. The heart of the Circle is the Weekly Math Circle lectures. Each week (typically a Sunday) during the academic year middle and high school students from San Francisco Bay would gather for two hours in the afternoon on a University campus (likely USF) for a lecture on a specific mathematical topic. University professors and teachers, well-known locally, nationally and even internationally for their expository and problem-solving skills, present the lectures. Frequently, the lectures take the form of an interactive discussion between the instructor and students, and the students volunteer to explain solutions and problems to the rest of the group. The style, organization, level and topic of the lectures varies from meeting to meeting. Some lectures are aimed at problem-solving for mathematical competitions. Other lectures introduce the students to exciting advanced math topics whose level range from elementary high school to advanced undergraduate. Yet, a third type of lecture deal with connections between mathematics and other sciences such as physics, biology, computer science, and economics.

San Francisco Math Circle

The Berkeley Math Circle does a good job of providing challenges for young people who are already motivated and intrigued by complex mathematical problems and ideas, and thus is not focused on developing that kind of interest and fascination. The San Jose Math Circle, together with its monthly program Bay Area Mathematical Adventures (in conjunction with the University of Santa Clara) is more focused on the seduction of mathematical conundrums, and appeals to a much wider (as well as younger audience). The San Francisco Math Circle will draw on the ideas and techniques of both circles, and will aggressively recruit students from San Francisco schools through a network of teachers and educational administrators yet to be identified. Both the Berkeley and San Jose Circles depend on this network to recruit student and teacher interest as well as instructors from the corporate world. In particular, we propose to take advantage of the wealth of talent existing in the financial, communications and computing communities of San Francisco. We will need four to six months lead time providing demonstrations in the schools and demonstrations in industry to develop these networks.

San Francisco Math Circle students would be welcomed and encouraged to participate in the annual Bay Area Mathematics Olympiad contest which is a funded and on-going program. BAMO is an exam given once a year to students at participating high schools and middle schools, most of whom are in the San Francisco Bay Area. The exams are mailed out to the schools, proctored locally, then returned to be graded

by a group of math circle instructors and educators. The following weekend, there is an awards ceremony, with prizes for individuals and schools, lunch for everybody, and a math lecture by a distinguished mathematician. This awards ceremony has become an annual focal point for the Bay Area middle and high school activities, where about 200 students, teachers and parents gather for an exciting day of Mathematics. The event has been made even more worthwhile by the involvement of famous lecturers and their fabulous talks.

Paul Zeitz, Professor of Mathematics at the University of San Francisco, and the coordinator of BAMO, has agreed to serve as director of the SFMC, in close coordination with Zvezdelina Stankova, Professor of Mathematics at Mills College and director of the BMC, and Tatiana Shubin, Professor of Mathematics at San Jose State University and director of the San Jose MC. We will contact faculty at San Francisco State University who have educational programs in the San Francisco School district to work with us in recruiting students and teacher support. We have had contact with Marian Currell of the M. L. King Middle School in connection with the Algebra Project; she will be an important resource in accessing minority populations and developing familiarity with middle school students. There is an already established pool of about 30 Bay Area mathematicians and professionals directing Math Circle sessions in Berkeley and San Jose who will also be used for the San Francisco MC.

Our plan is to spend the academic year 2004–2005 establishing and developing these contacts, formulating the program and setting up a website for the SFMC. We will run several pilot sessions in selected San Francisco schools in the Spring, and run several Saturday training sessions for teachers. Then, in September 2005 we will begin the first year of the SFMC, to be run in weekly two-hour sessions at the University of San Francisco.

Recognition

MSRI would be pleased to work with the underwriter of this project to develop awareness with the company's target audience of the company's philanthropic support and values. Media will be informed of the project and its corporate supporter, and will be invited to and assisted in the development of stories to publicize the program. MSRI will work with XYZ Corporation to publicize this program in internal communications also. Program printed materials and promotions will make mention of the company's support. A circle banner, prominently displaying XYZ Corporation and its sponsorship will be developed and displayed at weekly Circle meetings. Student participants and their parents will be made aware of the sponsor's identity at meetings and other gatherings. We would appreciate knowing more about XYZ Corporation's objectives in this area, and will be pleased to work with them to accomplish these objectives.

MSRI administers a pilot program promoting interest in mathematics with grade school students through bus posters and a related web site. Visitors to the website will be referred to the San Francisco Math Circle for more information and involvement with mathematics.

Evaluation

MSRI proposes to report each year the following measurable results to XYZ Corporation: List of Math Circle lectures with dates, topics, and name and credentials of lecturers; the number of participants in the program; the gender and stated race/ethnicity of the participants; the number of schools represented; the BAMO date, location; and the number of San Francisco Math Circle participants; the date, location and the number

of San Francisco Math Circle participants in national and international competitions; the known college affiliation of former participants; the known number of participants who go on to teach or do research in mathematics. Annual financial reports will be submitted on program budgets and actual annual expenditures. Special reports will be provided upon request.

Budget

2004–2005 Academic Year	
Circle Directors' Compensation (part-time)	\$10,000
Web Design and Support	\$3,500
Advertising and Announcements	\$1,000
Honoraria for Circle Instructors (6 pilot programs in SF schools)	\$1,200
Compensation for Teachers (two day sessions with 15 teachers each)	\$3,000
Publicity and Promotion	\$2,000
Administrative Support	\$6,000
TOTAL	\$26,700
2005–2006 Academic Year	
Circle Directors' Compensation (part-time)	\$24,000
Web Maintenance	\$1,000
Honoraria for Circle Instructors (32 sessions)	\$6,400
Compensation for Teachers (10 monthly sessions with 12 teachers each)	\$12,000
Publicity and Promotion	\$600
Administrative Support	\$12,100
TOTAL	\$56,100

Appendix D

Sample Grant Report

A wide range of mathematical activities in the San Francisco Bay Area are made possible by a generous grant from a particular foundation. In order to maintain the level of funding necessary to continue and expand these activities each year, it is crucial to convey to the foundation the positive impact that their support makes. The report appearing in the following pages was prepared by Brandy Wieggers to describe one year in the life of the newly formed Oakland/East Bay Math Circle, which is supported by this grant. Notice the attention to detail—after reading this document the donor has a very good sense of what the math circle has set out to accomplish, how successfully it has met its goals, who is responsible for its success, who is benefitting from the program, how the participants feel about their experience, and much more. It is a model document for anyone setting out to create their own grant report. (The report has been edited for length and formatting. Significant changes from the original are indicated by italicized remarks.)

Oakland/East Bay Math Circle 2007-2008 Final Report



The Oakland/East Bay Math Circle (OEBMC) offers middle school students and teachers in the Oakland and East Bay area the opportunity to learn advanced mathematics in a free, after-school enrichment program. The sessions take place on Mondays from 4:30–5:30 at Laney College in Oakland, preceded by a snack and social time. On the second Monday of each month a longer Teachers' Circle session is held to continue the work begun during the intensive summer Teachers' Circle held at MSRI.

Attendance

The summer Teachers' Circle program led to a healthy start of the OEBMC program at Laney College. An average of twenty students participate each week; a total of sixty-six students attended at some point during the year. These students represent eight area schools, five of which have teachers that are bringing students regularly to the program. For the Fall program there was a combined session for middle and high school students, along with a session for teachers. The Fall student session was such a success that we expanded it for the Spring to include a separate high school (more theory based) session and middle school (more activity based) session, to allow for more student participation. Consequently our high school attendance grew to an average of fifteen students per session towards the end of the year.

Program

Introducing topics from probability to population dynamics, the student session started strong with a series of sessions provided by some of the area's most enthusiastic graduate students. These instructors shared their own experiences as middle school math explorers and easily related to the students. One of the most popular sessions involved origami, with students bringing their completed folded works to later sessions to share. Teachers also investigated a wide variety of mathematical topics, with many

session instructors surprised by how little they had to teach because of the degree to which teachers helped each other explore the topics based on their own personal experience.

Role Models

Twenty-two mathematicians from the Bay Area were involved in the 2007–2008 OEBMC program. These professionals provided not only a series of informative and exciting presentations, but also a diverse set of mathematical role models, running the gamut from graduate student to dean. By happy coincidence the Fall instructors for the student sessions alternated female and male, a pattern of diversity that we will try to continue in future sessions.

Conclusion

This is an extraordinary program that is leaving the students and teachers wanting more. After a successful start to the program last Fall, we expanded the Spring program to include more sessions, monthly problem contests, and regular newsletters sent to students' home addresses. All of these additions helped to build up the program. As a result students, teachers, and staff alike are greatly anticipating the 2008–2009 Oakland/East Bay Math Circle, schedule to begin on September 18, 2008.

Data on Attendance, Diversity, Logistics

The following figures provide a quick overview of the OEBMC for the 2007–2008 academic year.

72 Total student participants in the program

21 Average weekly student participants

17 Total number of teachers and administrators invited for the week-long Summer Teachers' Circle Program

11 Total number of teachers and administrators who completed the week-long Summer Teachers' Circle Program

13 Total number of teachers and administrators who have attended the OEBMC Program

5.5 Average number of teacher and administrator participants (two teachers alternate weeks)

- 8 Total number of schools participating
 12 Total schools sending participants to Summer Training
 5 Total schools with teachers regularly bringing students
 8 Total number of schools represented by students

Student and Teacher Diversity Data

Visual Ethnic Composition of Students

Ethnicity	Mid-year	May
Asian	35%	47.0%
African-American	23%	18.1%
Latino/Hispanic	15%	11.1%
Caucasian	10%	8.3%
Unknown	17%	15.5%

Gender Ratio of Students

Gender	Mid-year	May
male	55%	56.9%
female	45%	43.1%

Grade Levels of Students

Grade Level	Mid-year	May
Middle school (6th–8th grade)	33 (82.5%)	42 (58.3%)
High school (9th–12th grade)	6 (15%)	29 (40.2%)
Elementary school (5th grade)	1 (2.5%)	1 (1.5%)

Gender Ratio of Teachers

Gender	Mid-year	May
male	36%	42%
female	64%	58%

Grade Levels Taught by Teachers

Grade Level	Mid-year	May
Middle school (6th–8th grade)	73%	67%
High school (9th–12th grade)	18%	25%
Elementary school (5th grade)	9%	8%

Schools Involved in the OEBMC

The following schools participated in the OEBMC during the past year. Schools with teachers who attended the summer program (TSP) and schools with teachers bringing students (TBS) are indicated following the school name.

Albany Middle School, Albany (TSP/TBS)
 Bay Area School for Independent Studies, Newark-Concord (TSP/TBS)
 Bay Farm Elementary School, Alameda
 Bret Harte Middle School, Oakland (TSP/TBS)
 Lincoln Middle School, Oakland
 Lionel Wilson Middle School, Oakland (TBS)
 Oakland Senior High School, Oakland (TSP/TBS)
 Willard Middle School, Berkeley (TSP/TBS)

Participants in the Summer Training

The following schools sent teachers to take part in the intensive week-long course at MSRI during the summer of 2007.

ACCESS program at Lawrence Hall of Science
 Bay Area School for Independent Studies, Newark
 Bret Harte Middle School, Oakland
 Bentley Middle School, Oakland
 Castilleja School, Palo Alto
 Elmhurst Community Prep, Oakland
 The Girls' Middle School, Mountain View
 Oakland Senior High School, Oakland
 Roosevelt Middle School, Oakland
 Santa Ana Opportunity School, Hollister
 Woodside Learning Center, San Francisco
 Willard Middle School, Berkeley

Recruitment Efforts

Summer Teachers' Circle Workshop

The primary recruitment used for the Oakland/East Bay Math Circle was the Summer Teachers' Circle workshop. By associating the Teachers' Circle and Math Circle we have been able to quickly develop our core group of participants. We look forward to the continued success of this recruitment method next year.

Recruitment Events

In addition to relying on the Summer Workshop teachers, we attended the following recruitment events in an effort to advertise and promote the program.

Julia Robinson Mathematics Festival at Pixar, Emeryville, May 3, 2008

East Bay Community Foundation Spring Mathematics and Science Forum, April 29, 2008

Joint MDTP/EAP/ Cal-Pass Conference for Math Teachers: Using Assessment Information for the Benefit of Students, March 12, 2008.

Oakland Unified School District Meeting of Secondary Mathematics Department Chairs, February 27, 2008

California Mathematics Council Conference, November 30 to December 1, 2007

Cal-Girls Mini Grant Conference, October 10, 2007

Oakland Unified School District Meeting of Secondary Mathematics Department Chairs, September 26, 2007

Retention Efforts

After school programs suffer from retention, a problem that is most obvious after a long program break (such as the three-month Winter break) or after a long-weekend. Anything that breaks the routine tends to decrease student participation. In order to retain students, the OEBMC program created a monthly program newsletter and problem contest. The newsletter announced the weekly scheduled, reminded students about upcoming events

and announced the monthly problem contest. By mailing these newsletters home and providing them at the weekly meetings, the program saw an overall increase in student retention. In addition, the students enjoyed the monthly problem contest, which provided math prizes (Zome construction tools) to the winner of the monthly problem raffle. The contest also proved to be useful in the classroom as a warm-up activity for those students who arrived early.

Oakland/East Bay Math Circle Program

As a representative sampling, we present here the spring session schedule for middle school students followed by the year-long schedule of math circles held for teachers. The original report also described the combined student fall session and the spring session for high school students.

Spring Session: Middle School Students

Date	Instructor	Topic
Feb 4	Hillel Raz	Hungry Hippo Problem
Feb 11	Ezra Gouvea	Introduction to Topology
Feb 25	Yaim Cooper	Greek Geometry
Mar 3	Hillel Raz	Cryptography
Mar 10	Hillel Raz	Cryptography
Mar 17	Ezra Gouvea	Abstraction and Topology
Apr 7	Yvonne Lai	The Porridge Portion Problem
Apr 14	Matt Kahle	Tilings
Apr 21	Alex Kaiser	Geometry and Minimization
Apr 28	Jana Comstock	Lewis Carroll Logic
May 5	Yvonne Lai	Tic-Tac-Toe on Mini Spaces
May 12	MSRI hosts	End of Year Party

Fall and Spring Sessions: Teachers

Date	Instructor	Topic
Sept 10	Tatiana Shubin	Math Games
Sept 17	Charles Crissman	Winning Game Strategies
Sept 24	Charles Crissman	Winning Game Strategies
Oct 1	Brandy Wieggers	Teachers Meeting
Oct 8	Joshua Zucker	One Player Games
Oct 15	Ezra Gouvea	Mobius Strips
Oct 22	Ezra Gouvea	Geometric Topology
Oct 29	Yaim Cooper	Definitions and Algorithms
Nov 5	Michael Orkin, David Ross	Laws of Chance and Opportunities for Math at Laney
Jan 14	Paul Zeitz	Applications of Number Theory
Feb 4	Matt Beck	Codes, Ciphers, Secret Messages
Feb 11	Tom Davis	Global Geometry
Feb 25	Peter Tingley	How Pool Balls Bounce
Mar 3	Matt Beck	Pick's Formula and Polygons
Mar 10	Tatiana Shubin	Problems to Tease and Teach
Mar 17	Tom Rike	Mass Point Geometry
Apr 7	Emily Peters	Mathematics of Origami
Apr 14	Joshua Zucker	Pigeon Hole Principle
Apr 21	Yvonne Lai	The Porridge Portion Problem
Apr 28	Yaim Cooper	Euler Characteristic
May 5	Matt Beck	Parking Functions
May 12	MSRI hosts	End of Year Party

Session Speaker Information

Matt Beck, faculty at San Francisco State University and teacher for San Francisco Math Circle.

Jana Comstock, grad student, University of California, Berkeley.

Yaim Cooper, grad student, University of California, Berkeley.

APPENDIX D. SAMPLE GRANT REPORT

213

Charles Crissman, graduate student at University of California, Berkeley.

Tom Davis, mathematician and director of San Jose Math Circle.

Jonathan Dorfman, previous speaker for Berkeley Math Circle.

Ezra Gouvea, graduate student at University of California, Davis.

Matt Kahle, postdoctoral fellow at Stanford University.

Alex Kaiser, undergraduate student, University of California, Berkeley.

Yvonne Lai, graduate student at University of California, Davis.

Michael Orkin, Dean of Business, Mathematics, and Sciences at Laney College.

Emily Peters, grad student at University of California, Berkeley.

Hillel Raz, graduate student at University of California, Davis and director of University of California, Davis Math Circle.

Tom Rike, previous Oakland Unified School District High School teacher and Berkeley Math Circle speaker.

David Ross, instructor and Chair of Mathematics Department at Laney College.

Tatiana Shubin, OEBTC Director, San Jose Math Circle Director, and faculty at San Jose State University.

Peter Tingley, grad student at University of California, Berkeley.

Brandy Wieggers, OEBMC Director, graduate student at University of California, Davis.

Ellen Veomett, faculty at California State University, East Bay

Joshua Zucker, OEBTC Director, faculty at Castilleja School.

OEBMC 2007-2008 Evaluation Summary

Student evaluations were given out on the last day of the Math Circle. The summary calculates the answers and provides quotes, as written by the students. No changes were made to the students' grammar, spelling or punctuation. A total of thirty evaluations were collected.

All the questions appearing on the evaluation are reproduced below. However, quotes from students are provided for only a couple of the questions. The original report contained a much more comprehensive set of responses.

- What grade are you in?
- What school do you attend?
- Do you plan to attend next year? If no, explain why not.

YES: 20 NO: 4 MAYBE: 6

- Describe your overall experience with the OEBMC program in two sentences.

I came very late into the school year so I did not learn everything I could from this program.

I learn about symmetry. I learn some stuff about geometry.

I liked some because it was interesting and dislike others.

It was fun coming even though this is my second time coming.

I had the most fun here. I also learned a lot of things.

It was great. And I had a lot of fun learning math.

It was a fun and entertaining experience.

It is interesting. It is educational.

My overall experience here has been great. I learned alot of stuff that I had forgotten.

It was fun being with my friends. I learned a few things.

- What did you like about the program?
- What things should we do again next year? What could we do better next year?

You did everything good. I loved making bubbles.

Everything

Stay the same.

The paper origami and airplane contests.

We should do the paper airplane contest again & you couldn't do anything better, because its already good!

Make the lessons a little more fun.

Make the lessons more hands on. The bubbles were fun.

I'm not sure.

Do everything again next year.

The same. I think everything is good.

Maybe more work time.

Have Ezra teach again.

More hands on stuff! and I liked the fact we had snacks and the time was nice too,

Making the lessons longer.

You should provide water.

You should teach stuff that was above our grade level.

All the stuff I learned was review from my previous math class.

Be more creative on the subject

Activities. Better scheduling of programs.

Bubbles

I think it is pretty good now and not having been here that long I'm not sure what should be change.

I don't know.

The problem of the month

Nothings, it's at it's best

- What did you gain from this program? (Check all that apply) Information/perspective, new skills, learned about new math, new friends, emotional support, extra credit at my school, inspiration/motivation, commitment to personal change.
- What did you enjoy more? Circle one of the two options given. Working together as a group vs working as an individual, working on math problems vs learning about mathematical theories, activities vs lecture, snacks vs prizes. (Responses from middle school students and high school students are tallied separately.)

	MS	HS	Total	Percent
Group	7	15	22	73%
Individual	3	3	6	20%
Math problems	6	10	16	53%
Math theories	5	7	12	40%
Activities	9	15	24	80%
Lecture	3	2	5	17%
Snacks	7	16	23	77%
Prizes	5	4	9	30%

- What are you going to do after you graduate from high school? Have your experiences in this program affected your answer for this (and if so, how)?
- If you were in college now, what major would you choose? Have your experiences in this program affected your answer for this (and if so, how)?
- How has this program affected the way you perceive math?

Students were then asked to rate each of the sessions that they attended throughout the year. Based on student responses, an average rating (from 1–7) was reported for each of the math circles, along with any remarks from students.

The report concluded with two full-page color picture collages, one showcasing student activities and the other featuring the teachers.

Acknowledgments

We gratefully acknowledge the kindness of the following institutions and individuals in granting the following permissions:

Diagram of stereographic projection in Chapter 14

D. Joyce, Clark University

Photograph of Robert and Ellen Kaplan

Courtesy of Robert Kaplan and Ellen F. Kaplan

Photograph of Vasiliy Prokhorov

Courtesy of Vasiliy Prokhorov

Photograph of Richard Rusczyk

Courtesy of Richard Rusczyk

Photograph of Tatiana Shubin and Tom Davis

Courtesy of Tatiana Shubin and Thomas R. Davis

Photograph of Zvezdelina Stankova

Courtesy of Zvezdelina Stankova

Photograph of Jim Tanton

Courtesy of James Tanton

Photograph of Blake Thornton

Courtesy of Blake Thornton

Photograph of Sam Vandervelde

Courtesy of Samuel K. Vandervelde

Photograph of Paul Zeitz

Courtesy of Paul Zeitz

Math circles provide a setting in which mathematicians work with secondary school students who are interested in mathematics. This form of outreach, which has existed for decades in Russia, Bulgaria, and other countries, is now rapidly spreading across the United States as well. The first part of this book offers helpful advice on all aspects of math circle operations, culled from conversations with over a dozen directors of successful math circles. Topics include creative means for getting the word out to students, sound principles for selecting effective speakers, guidelines for securing financial support, and tips for designing an exciting math circle session. The purpose of this discussion is to enable math circle coordinators to establish a thriving group in which students can experience the delight of mathematical investigation. The second part of the book outlines ten independent math circle sessions, covering a variety of topics and difficulty levels. Each chapter contains detailed presentation notes along with a useful collection of problems and solutions. This book will be an indispensable resource for any individual involved with a math circle or anyone who would like to see one begin in his or her community.



Sam Vandervelde teaches at St. Lawrence University. He launched the Stanford Math Circle and also writes and coordinates the Mandelbrot Competition, a math contest for high schools.

ISBN 978-0-8218-4752-7



9 780821 847527

MCL/2



For additional information
and updates on this book, visit

www.ams.org/bookpages/mcl-2

AMS on the Web
www.ams.org